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MILLIAN EFFICIENCY WITH ENDOGENOUS FERTILITY

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Millian Efficiency with Endogenous Fertility*

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Abstract

This paper is concerned with an extension of the notion of Pareto efficiency, referred to as *Millian efficiency*, to evaluate the performance of symmetric allocations in an overlapping generations setting with endogenous fertility. The criterium of Pareto dominance underlying the notion of Millian efficiency is based exclusively on preferences of those agents who are actually born, and allows only for welfare comparisons of symmetric allocations (i.e, allocations in which all living individuals of the same generation take the same decisions). The main contributions of the paper are the following. First, we provide necessary (static) and sufficient (dynamic) conditions to determine whether an allocation is Millian efficient or not, and we show that the sufficient conditions for dynamic efficiency offered by Cass (1972) and Balasko and Shell (1980) cannot be straightforward applied when fertility decisions are endogenous. Second, we extend the two Fundamental Theorems of Welfare Economics to a framework with endogenous population by characterizing Millian efficient allocations as the equilibria of a decentralized price mechanism. Finally, we present a condition to identify equilibrium allocations as dynamically efficient that exclusively uses the sequence of prices associated to such decentralized equilibria.

Keywords: Endogenous fertility, Pareto optimality, Dynamic efficiency.
JEL: D61, D91, H21, J13

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1 Introduction

This paper studies a notion of efficiency, which we refer to as *Millian efficiency*, applicable to evaluate equilibrium allocations in an overlapping generations framework with endogenous fertility decisions and capital accumulation. In our model: (i) all agents of a given generation have the same preferences on consumption bundles, represented by a well behaved utility function; (ii) children are a costly consumption good, and parents derive utility from the number of children they bear, but not from the utility of their descendants and, (iii) fertility choices are selected from a continuum. In this setting, the notion of Millian efficiency results from combining:

a) an extension of the Pareto dominance criterium, referred to in the literature as the $\mathcal{A}-$dominance criterium, which compares any two allocations of different population size by comparing exclusively the welfare profiles of those agents who are alive in the two allocations; and,

b) a constraint on the set of allocations that can be compared using the $\mathcal{A}-$dominance criterium, which is restricted to be formed by all feasible allocations in which i) every two living agents of the same generation are treated equally and therefore obtain the same consumption bundles; and, ii) the population size of each generation is strictly positive.

To be more precise, a Millian efficient allocation is a symmetric allocation with positive fertility rates for every period that is not $\mathcal{A}-$dominated by any other symmetric allocation that also yields positive fertility rates. As it is shown in the paper, restricting welfare comparisons to allocations with strictly positive fertility rates is necessary in order to have a well defined efficiency criterium, since otherwise the $\mathcal{A}-$dominant criterium induces a non-transitive relation on the set of feasible allocations. Despite other names (such as, for example, *constrained* $\mathcal{A}-$*efficiency*) might be more informative of the normative principles underlying this notion of efficiency, we use the term Millian efficiency because it generalizes a notion of optimality, referred to as *Millian optimality*. This criterium might be regarded as a form of utilitarianism, called average utilitarianism, often associated to John Stuart Mill, which postulates that welfare judgments involving different generations should be independent of the population size of each generation.[1]

Once we adopt this extension of the notion of Pareto efficiency, we explore its properties in the framework studied in the paper. First, we obtain a set of conditions that are necessary for achieving *statically efficient,* i.e., an allocation that cannot be improved upon by a reallocation of resources of a finite number of generations. Second, we provide sufficient conditions guaranteeing that a statically efficient allocation is in fact efficient (or *dynamically efficient*): a statically

[1]See Razin and Sadka, 1995, ch.5). See also Nerlove, Razin and Sadka (1982), Cigno (1992, 2003) or Groezen, Leers and Medjam (2003).

efficient allocation that might not be improved upon a reallocation of resources involving an infinite sequence of intergenerational transfers. Some authors have presented necessary and sufficient conditions for Pareto efficiency in different overlapping generation settings, but all of them consider fertility as exogenous. The best known criterium to determine whether or not a stationary path is dynamically efficient with exogenous fertility was found by Phelps (1965), Koopmans (1963) and Diamond (1965) in different settings, and imposes that the long run interest rate exceeds the rate of population growth. Other well known results are those by Cass (1972), who provided the first complete characterization of dynamically efficient paths in a production economy, or Balasko and Shell (1980), who studied dynamic efficiency of exchange economies in an overlapping generations model.

As we show in the paper, considering fertility as an endogenous decision introduces non convexities on the sequence of inequalities characterizing the set of feasible allocations, because some aggregate variables are the product of two endogenous variables. Due to these non-convexities, standard criteria based on the ratio of the long run interest rate to the rate of population growth are no longer valid to identify efficient paths. In view of this, we provide a sufficient condition for efficiency in non-convex settings.

Next, endowed with the tools necessary to determine what fertility choices might be considered efficient, we explore under what conditions decentralized decisions lead to efficient choices. To do this, we adapt the Fundamental Theorems of Welfare Economics to a setting with endogenous fertility by characterizing every (statically) Millian efficient allocation as the equilibrium of a decentralized sequential price mechanism. Analogous to the case of economies with exogenous population, every Millian efficient allocation can be decentralized by initially selecting an appropriate sequence of intergenerational transfers, and by allowing then the agents to determine their consumption and investment decisions at competitive markets. Differently from the standard exogenous population case, in which non-distorting intergenerational transfers must be lump-sum for all agents, an incentive scheme that links intergenerational transfers with fertility decisions is needed. More precisely, for every system of intergenerational transfers that achieves Millian efficiency, every middle-aged adult has to pay a lump-sum tax (or, in some cases, receive a lump-sum subsidy), while every old adult has to receive a subsidy (or pay a tax) which depends linearly of the number of children she decided to have. As a particular case, we also show that the allocation corresponding to a decentralized equilibrium with no intergenerational transfers (for which there is no need to subsidize or tax children) is (statically) Millian efficient. In contrast with other environments with incomplete markets, this particular case shows that the absence of a market (in this case, a market where offspring may bargain with their parents the right to be born) does not yield any efficiency loss, at least if one is concerned with Millian efficiency.

To conclude the paper, we find a simple criterium to identify dynamically efficient paths for any competitive decentralized equilibrium. We show that the standard criterium to define dynamic efficiency for the case of exogenous fertility decisions should be replaced by an alternative criterium, that states that the rate of return to physical capital should be higher than the highest rate of return to invest in children.

In the literature of overlapping generations economies with endogenous fertility, two different approaches to provide normative principles can be distinguished: a first approach identifies socially optimal allocations with steady state optimal allocations (also referred to as golden rule allocations), that is, allocations that maximize the utility obtained by a representative consumer among those feasible stationary allocations;[2] while a second approach identifies optimal allocations with those maximizing a certain class of social welfare maximization problems, referred to as Millian or Benthamite depending on whether or not the welfare weight given to a generation in the social welfare function depends on the size of that generation.[3] Neither one of these two approaches takes explicitly into account the problem of dynamic efficiency, nor the fact that the standard Pareto criterium is not straightforward applicable to environments in which the set of agents is endogenous.

An interesting exceptions within the literature of endogenous fertility are the papers by Michel and Wigniolle (2003) and Golosov, Jones and Tertilt (2004). Golosov *et al* consider a general overlapping generations economy in which fertility decisions are discrete, and assume that all potential agents –included those that will never be born– have well defined preferences. In this context, they analyze two extensions of the Pareto-dominance criterium. However, their assumption of a discrete set of potential agents brings with it considerable difficulties if one is concerned with identifying efficient allocations in overlapping generations settings with non-altruistic agents, as the one studied in this paper. Besides, the dynamic efficient problem do not arise in their framework because individual agents are linked intertemporally in such a way that their individual problems can be jointly solved by an infinitely-lived dynastic family problem. Michel *et al* have also proposed a notion of efficiency, referred to as Pareto optimality, which coincides with our notion of Millian efficiency, although it is not explicitly deduced from an extension of the Pareto criterium as it is in this paper. In addition, their treatment of the dynamic efficiency problem is substantially less general, since they restrict the analysis to equilibrium paths –with no intergenerational transfers– that converge to stationary states, and making use CES utility and production functions.

[2]See e.g., Samuelson (1975, 1976), Deardoff (1976), Eckstein and Wolpin (1985), Bental (1989) or Michel and Pestieau (1993).

[3]See e.g., Nerlove, Razin and Sadka (1982, 1985), Cigno (1993), and Groezen, Leers and Medjam (2003), or Razin and Sadka (1995, Ch.5) for a survey.

The paper is organized as follows. In section 2, we introduce the model. Next, in section 3, we present the notion of Millian efficiency and provide necessary and sufficient conditions to determine whether an allocation is efficient in this sense. In section 4 we characterize Millian efficient allocations as the equilibria of a decentralized sequential price mechanism. Finally, section 5 presents the main conclusions of the paper and discusses possible extensions.

2 The Model: Assumptions and Definitions

Consider an overlapping generations economy with three generations of consumers (referred to as *old*, *middle-aged* and *children)* coexisting at each period $t = 0, 1, 2, ...$ In each period t there exist $N_{t-2} \in \Re_+$ old adults (who were born at date $t-2$), N_{t-1} middle-aged adults (born at date $t-1$) and $N_t \in \Re_+$ children (born at t). For each $t = 0, 1, 2, ...$,write

$$n_t = \begin{cases} \frac{N_t}{N_{t-1}}, & \text{if } N_{t-1} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The number of old adults at $t = 0$ is normalized to one, and the number of middle-aged adults at $t = 0$ is given by the initial condition $N_{-1} = \overline{n}_{-1} > 0$.

Resources available can be described as follows. Middle-aged agents are endowed with one unit of time to work, which is supplied inelastically. At each period $t = 0, 1, 2...$, a perishable consumption good is produced using labor (N_{t-1}) and physical capital (K_t) invested in previous period $t-1$ as inputs, that is,

$$Y_t = F_t(K_t, N_{t-1}),$$

where Y_t is total output, and $F_t : \Re^2_+ \to \Re$ is a differentiable concave and constant return to scale production function. Physical capital is fully depreciated in the production process, and the stock of capital at period $t = 0$ is given by the initial condition $K_0 = \overline{K}_0$.

Throughout the paper, we will restrict attention to symmetric allocations in which any two agents of the same generation who get to be alive take the same consumption and investment decisions. The aggregate output of the homogeneous good is used to finance aggregate investments in capital, denoted by K_{t+1}, to finance aggregate consumption by old adults (denoted by C^o_t) and by middle-aged adults (denoted by C^m_t), and to cover costs of rearing children. Rearing children is a production activity that takes place within each household and its costs (per middle aged adult of every family) depend on the state of technology and the number of children raised within the family. More precisely, costs per middle aged adult of raising n_t children at time t are given by $b_t(n_t)$, where b_t is a differentiable non-decreasing, and convex function satisfying $b_t(0) = 0$.

At any period, the aggregate resource constraint is

$$C^o_t + C^m_t + N_{t-1}b_t(n_t) + K_{t+1} \leq F_t(K_t, N_{t-1}), \tag{1}$$

which for allocations exhibiting positive fertility rates at every period can be equivalently written as

$$c_t^o + n_{t-1}\left[c_t^m + b_t(n_t) + k_{t+1}^o\right] \leq F_t(k_t^o, n_{t-1}), \tag{2}$$

where $c_t^o = C_t^o/N_{t-2}$ represents average consumption per old adult, $c_t^m = C_t^m/N_{t-1}$ represents average consumption per worker, and $k_t^o = K_t/N_{t-2}$ represents capital invested per old adult.

A feasible symmetric allocation will be represented by a sequence $a \equiv \{a_t\}_{t=0}^{\infty} = \{(c_t^m, c_t^o, n_t, k_t^o)\}_{t=0}^{\infty}$ satisfying, for each $t = 0, 1, 2...$, the resource constraint in (1) and the initial condition

$$(n_{-2}, n_{-1}, k_0^o) = \left(1, \overline{n}_{-1}, \overline{K}_0\right). \tag{3}$$

Denote by $\mathcal{S}$ the set containing all feasible symmetric allocations. For each agent born at $t = -2$, preferences on $\mathcal{S}$ are represented by a utility function $U_{-2} : \mathcal{S} \to \Re$ defined, for each $a \in \mathcal{S}$, by $U_{-2}(a) = c_0^o$, where c_0^o denotes the old adult's consumption at period $t = 0$. For each agent born in period $t-1$ with $t = 0, 1, 2, ...$, preferences are represented by a utility function $U_{t-1} : \mathcal{S} \to \Re$ defined, for each $a \in \mathcal{S}$, by $U_{t-1}(a) = u(x_t) = u(c_t^m, c_{t+1}^o, n_t)$, so that individuals may receive direct utility from consumption as well as the number of descendants they bear. We assume that the function $u : \Re_+^3 \to \Re$ is differenciable, continuous, strictly increasing and quasiconcave on $\Re_{++}^3$.

Two final observations are in order. First, notice that this representation imposes that agents cannot obtain resources for their old without having children, i.e., for every allocation for which $n_{t-1} = 0$ one necessarily has $y_t^o = c_t^o = 0$. Thus, such representation seems inadequate to represent allocations for which $n_{t-1} = 0$ for some period t in environments where $F_t(K_t, 0) \neq 0$ for some $K_t > 0$. Finally, note that the term $n_{t-1}\left[c_t^m + b_t(n_t) + k_{t+1}^o\right]$ in the left hand side of (2) is a quasiconcave function of the endogenous variables n_{t-1}, c_t^m, n_t and k_{t+1}^o. Due to this fact, the set of sequences $a = \left\{\left(c_t^m, c_t^o, n_t, k_{t+1}^o\right)\right\}_{t=0}^{\infty}$ satisfying the resource constraint in (2) and the initial condition (3) is not a convex set, as it would if the sequence $\{n_t\}_{t=0}^{\infty}$ were fixed exogenously.

3 Millian Efficient Allocations

The most commonly used optimality notion in standard normative economic analysis is that of Pareto efficiency. This notion of efficiency relies in turn on the well known Pareto criterium to compare social alternatives, a criterium that allows one to construct a *partial ordering* on the set of alternatives from the *complete preference orderings* (defined on such consumption set) of a fixed group of agents. An efficient allocation can be described as a maximal element of the partial order induced by the Pareto criterium on the set of feasible allocations.

With endogenous populations, we can still use the Pareto criterium to rank feasible allocations using the partial orderings of all potential agents, represented by the utility functions of the living agents. That is, an allocation can still be ranked as Pareto superior to another one if it is unanimously preferred by all potential agents according to their partial preference ordering. However, this implies that any two allocations with different population size cannot be ranked, since we do not know whether or not an agent who lives in one allocation a but not in other allocation a' is better off in the latter than he is in the former. To avoid this problem and preserve the partial order induced by the Pareto criterium, one needs to extend it to compare also allocations of different population size.

A possible general extension of the Pareto criterium, applicable to any environment with endogenous fertility, can be constructed by ranking any two allocations making use of the Pareto criterium when the information of the preference profiles of those agents who are born in the two allocations is considered. This extension has recently been used also by Golosov, Tertilt and Jones (2004), who refer to it as the $\mathcal{A}-$*dominance* criterium (where $\mathcal{A}$ stands for alive agents). More precisely, the notion of $\mathcal{A}-$*dominance* can be defined as follows.

Definition 1 *For any two feasible allocations* $\mathbf{a}, \mathbf{a}'$ *corresponding to an environment with endogenous fertility,* $\mathbf{a}$ *is said to* $\mathcal{A}-$*dominate an allocation* $\mathbf{a}'$ *if* $\mathbf{a}$ *is unanimously preferred to* $\mathbf{a}'$ *by all agents who are born in both* $\mathbf{a}$ *and* $\mathbf{a}'$*, and it is strictly preferred by some of these agents.*

Observe that the $\mathcal{A}-$dominance criterium is a general criterium, applicable to any two feasible allocations corresponding to an environment with endogenous population. The criterium can therefore be applied to rank any pair allocations. To avoid notational costs and make the $\mathcal{A}-$dominance criterium suitable to undertake welfare comparisons of symmetric allocations without specifying the identity of every potential agent, we will adopt in what follows the following convention: for every two symmetric allocations a, $a' \in \mathcal{S}$ for which the size of a given generation t is strictly positive (that is, such that $N_t > 0$ and $N_t' > 0$), there exists a positive measure of agents born at t in the two allocations.

With this convention, the restriction of the relation induced by the $\mathcal{A}-$dominance criterium to the set of symmetric allocations $\mathcal{S}$ can be defined formally as follows.

Definition 2 *A feasible allocation* $a \in \mathcal{S}$ *is said to* $\mathcal{A}-$*dominate an allocation* $a' \in \mathcal{S}$ *if*

i) for every $t = 0, 1, 2, ...$ *for which* $n_t > 0$ *and* $n_t' > 0$ *one has*

$$U_t(a) \geq U_t(a');$$

and,

ii) there exists at least one period τ *satisfying*

$$n_\tau \;>\; 0,$$

$$\begin{aligned} n'_\tau &> 0, \ \ and \\ U_\tau(a) &> U_\tau(a'). \end{aligned}$$

Thus, according to the $\mathcal{A}-$dominance criterium, a symmetric allocation a dominates another one a' if it provides all agents living under the two allocations with at least the same welfare, and some of them with more utility.

The following example shows that even if we restrict its scope to compare only symmetric allocations, the notion of $\mathcal{A}-$dominance brings with it an important difficulty: it induces a non-transitive relation on $\mathcal{S}$.

Example 1. *Non transitivity of the $\mathcal{A}-$dominance relation.* Consider a stationary economy described by a constant utility function $u(x_t) = 2\left(c_t^m\right)^{1/2} + 2\left(c_{t+1}^o\right)^{1/2} + \left(n_t\right)^{1/2}$, a constant production function $F_t(k_t, n_{t-1}) = 2k_t + 2n_{t-1}$, and a constant cost function $b_t(n_t) = n_t$. Observe that the stationary allocation $a = \left\{\left(c_t^m, c_{t+1}^o, n_t, k_{t+1}^o\right)\right\}_{t=0}^\infty$ such that $\left(c_t^m, c_{t+1}^o, n_t, k_{t+1}^o\right) = (1,1,1,1)$ for all $t \geq 0$ gives all agents who get to be alive a utility level $U_{-2} = 1$ and $U_{t-1}(a) = 5$ for all $t \geq 0$.

Consider now a date $\tau > 0$ and an allocation $\widetilde{a} = \left\{\left(\widetilde{c}_t^o, \widetilde{c}_t^m, \widetilde{n}_t, \widetilde{k}_{t+1}^o\right)\right\}_{t=0}^\infty$ such that

$$\left(\widetilde{c}_t^m, \widetilde{c}_{t+1}^o, \widetilde{n}_t, \widetilde{k}_{t+1}^o\right) = \begin{cases} (1,1,1,1), & \text{if } t = 0, 1..., \tau - 1 \\ (2,2,0,1), & \text{if } t = \tau \\ (0,0,0,0), & \text{if } t > \tau. \end{cases}$$

Such allocation yields $U_{-2} = 2$ and

$$U_{t-1}(\widetilde{a}) = \begin{cases} 5, & \text{if } t = 0, 1..., \tau - 1; \\ 4\sqrt{2} > 5, & \text{if } t = \tau; \end{cases}$$

which taking into account that only those agents born at $t \leq \tau$ are alive in both $\widetilde{a}$ and a, implies that $\widetilde{a}$ $\mathcal{A}-$dominates a. But then let $\overline{a} = \left\{\left(\overline{c}_t^o, \overline{c}_t^m, \overline{n}_t, \overline{k}_{t+1}^o\right)\right\}_{t=0}^\infty$ be an allocation such that

$$\left(\overline{c}_t^m, \overline{c}_{t+1}^o, \overline{n}_t, \overline{k}_{t+1}^o\right) = \begin{cases} (1,1,1,1), & \text{if } t = 0, 1..., \tau - 1 \\ \left(\frac{3}{2}, 3, \frac{1}{2}, 1\right), & \text{if } t = \tau \\ (0,0,0,0), & \text{if } t > \tau. \end{cases}$$

Such allocation yields $U_{-2} = 2$ and

$$U_{t-1}(\overline{a}) = \begin{cases} 5, & \text{if } t = 0, 1..., \tau - 1 \\ \sqrt{6} + 2\sqrt{3} + \sqrt{\frac{1}{2}} > 4\sqrt{2}, & \text{if } t = \tau \\ u(0) = 0, & \text{if } t = \tau + 1 \end{cases}$$

Hence, $\overline{a}$ $\mathcal{A}-$dominates $\widetilde{a}$. However, it is not the case that $\overline{a}$ $\mathcal{A}-$dominates a, since $\overline{n}_\tau > 0$ and $U_\tau(\overline{a}) = 0 < U_\tau(a) = 5$. Therefore the notion of $\mathcal{A}-$dominance induces a non-transitive relation on $\mathcal{S}$.□

It should be noticed that the type of inconsistency appearing in the previous Example 1 is present only if the $\mathcal{A}-$dominance criterium is used to compare allocations (like the allocation $\widetilde{a}$ in the example) for which the economy collapses at a given date and no more individuals are born. If this type of allocations are ruled out as socially undesirable and one restricts the set of allocations that can be compared to the set $\mathcal{S}^*$ formed by symmetric allocations $a \in \mathcal{S}$ such that $n_t > 0$ for all $t \geq 0$, such inconsistencies are no longer present. Observe that an allocation $a \in \mathcal{S}^*$ $\mathcal{A}-$dominates an allocation $a' \in \mathcal{S}^*$ if for all for all $t = 0, 1, 2, ...$ one has $U_t(a) \geq U_t(a')$ and this inequality is strict for some period t. Thus, the restriction of the $\mathcal{A}-$dominance relation to the set $\mathcal{S}^*$ is transitive and anti-symmetric, and therefore constitutes a partial ordering on $\mathcal{S}^*$.

With this restriction, the $\mathcal{A}-$dominance criterium gives rise to an efficiency criterium, which we refer to as *Millian efficiency* (or simply, $\mathcal{M}-$efficiency), to identify the set of maximal elements of the partial order induced by the $\mathcal{A}-$dominance criterium on the set $\mathcal{S}^*$. Formally, the notion of Millian efficiency can be defined as follows.

Definition 3 *A feasible allocation $\widehat{a} \in \mathcal{S}^*$ is said to be Millian efficient if there does not exist another feasible allocation $a' \in \mathcal{S}^*$such that:*
i) for all $t = -1, 0, 1, 2, ...$ one has

$$U_{t-1}(a') \geq U_{t-1}(\widehat{a}); \text{ and}$$

ii) there exists at least one period τ such that

$$U_{\tau-1}(a') > U_{\tau-1}(\widehat{a}).$$

Therefore our notion of Millian efficiency results from combining two elements. First, the direct extension of the Pareto dominance criterium to compare population of different sizes, the $\mathcal{A}-$*dominance criterium,* defined only through comparisons among agents, who are born. Second, the set of the allocations that can be compared is constraint to: *i)* symmetric allocations (i.e. every two living agents of the same generation are treated equally); and, *ii)* the population size of each generation is strictly positive.[4]

Thus, if an allocation is Millian efficient, then there is no way to make all living agents of every generation better off without making some living agents of a generation worse off. Although some other authors (i.e. Raut, 1992, and more recently, Michel and Wigniolle, 2003) have also used this criterium under the name of "Pareto optimality,"we use a different name to make clear that it results from *restricting* a particular *extension* of the Pareto criterium to the set $\mathcal{S}^*$ of feasible

[4]In Conde-Ruiz *et al* (2004, Sec.5), we show that if one keeps the symmetry restriction on the set of feasible allocations, then Millian efficient allocations may also be efficient under an alternative extension of the Pareto criterium obtained when an explicit specification of the preferences of non-born agents is introduced.

allocations. Also, although other names (such as, for example, $\mathcal{S}^*-$*constrained* $\mathcal{A}-$*efficiency*) might be more informative of the normative principles underlying this notion of efficiency, we use the term Millian efficiency because it generalizes a notion of optimality, referred to as *Millian optimality*, which has been frequently used in the literature. Finally, note that a Millian optimum is an allocation maximizing a function of the form $\mathcal{V}_\lambda(a) = \sum_{t=-1}^{\infty} \lambda_{t-1} U_{t-1}(a)$ for a strictly positive sequence of intergenerational weights $\lambda = \{\lambda_\tau\}_{\tau \geq -2}$. Clearly, a Millian optimum must be a Millian efficient allocation, but the converse is not in general true. Since the set of feasible allocations is unbounded, Millian social welfare functions may be not well defined for many feasible paths, including some that are Millian efficient ones.

In the following sections, we provide necessary and sufficient conditions characterizing Millian efficient allocations.

3.1 Necessary conditions. Static $\mathcal{M}-$efficiency.

For every allocation $a \in S$ and every $t \geq 0$, write e_t for the amount of physical resources at period t not devoted to feed the old generation, that is,

$$e_t = c_t^m + b_t(n_t) + k_{t+1}^o.$$

With this notation, the necessary conditions for Millian efficiency can be stated as follows.

PROPOSITION 1 *Every $\mathcal{M}-$efficient allocation $\widehat{a} \in \mathcal{S}^*$ verifies for each $t \geq 0$,*

$$u(\widehat{x}_t) = \max_{(x_t, k_{t+1}^o) \in \Re_+^4} \Big\{ u(x_t) \ : \ c_t^m + b_t(n_t) + k_{t+1}^o \leq \widehat{e}_t; \qquad F_{t+1}(k_{t+1}^o, n_t) - c_{t+1}^o \geq n_t \widehat{e}_{t+1} \Big\} = W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1}). \tag{4}$$

Proof. By contradiction. Suppose that $\widehat{a}$ is an $\mathcal{M}-$efficient allocation, and suppose there exists a period τ for which the 4-upla $(\widehat{x}_\tau, \widehat{k}_{\tau+1}^o)$ corresponding to the allocation $\widehat{a}$ is not a solution to the optimization problem in (4). Select now a point $(\widetilde{x}_\tau, \widetilde{k}_{\tau+1}^o) \in \Re_+^4$ satisfying the two constraints in (4) in such a way that $u(\widetilde{x}_t) > u(\widehat{x}_t)$ is satisfied, and let $\widetilde{a}$ be the allocation obtained from $\widehat{a}$ by replacing the term $(\widehat{x}_\tau, \widehat{k}_{\tau+1}^o)$ by such point. Such allocation is feasible because $(\widetilde{x}_\tau, \widetilde{k}_{\tau+1}^o)$ must verify $\widetilde{c}_t^m + b_t(\widetilde{n}_\tau) + \widetilde{k}_{\tau+1}^o \leq \widehat{e}_\tau$ and $F_{t+1}(\widetilde{k}_{t+1}^o, \widetilde{n}_t) - \widetilde{c}_{t+1}^o \geq \widetilde{n}_t \widehat{e}$. Note that $\widetilde{a}$ has been constructed in such a way that it satisfies $U_{t-1}(\widetilde{a}) = U_{t-1}(\widehat{a})$ for all $t \neq \tau$ and $U_{\tau-1}(\widetilde{a}) = u(\widetilde{x}_\tau) > u(\widehat{x}_\tau) = U_{t-1}(\widehat{a})$ for $t \neq \tau$, which implies that $\widehat{a}$ is not Millian efficient, a contradiction that establishes Proposition 1.$\square$

Since the preferences and technologies are differentiable, an interior solution $(x_t(e_t, e_{t+1}),$ $k_{t+1}(e_t, e_{t+1})) >> 0$ to the optimization problem in (4) is characterized by its first order conditions, given by:

$$\frac{u_1'(\widehat{x}_t)}{u_2'(\widehat{x}_t)} = D_1 F(\widehat{k}^o_{t+1}, \widehat{n}_t); \tag{5}$$

$$\left[b_t'(\widehat{n}_t) - \frac{u_3'(\widehat{x}_t)}{u_1'(\widehat{x}_t)}\right] \frac{u_1'(\widehat{x}_t)}{u_2'(\widehat{x}_t)} = D_2 F(\widehat{k}^o_{t+1}, \widehat{n}_t) - \widehat{e}_{t+1}; \tag{6}$$

and the two feasibility constraints

$$\widehat{c}^m_t + b_t(\widehat{n}_t) + \widehat{k}^o_{t+1} = \widehat{e}_t \tag{7}$$

and

$$F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_t) - \widehat{c}^o_{t+1} = \widehat{n}_t \widehat{e}_{t+1}. \tag{8}$$

Thus, for every interior, Millian efficient allocation, marginal rates of substitution between current and future consumption must be equal to marginal return to investments in physical capital. Note also that marginal willingness to pay for children (given by $u_3'(\widehat{x}_t)/u_1'(\widehat{x}_t)$) is not necessarily equal to marginal costs of rearing children; in this case, marginal rate of return to investments in children must be equal to the rate of return to any other investment:

$$\frac{D_2 F(\widehat{k}^o_{t+1}, \widehat{n}_t) - \widehat{e}_{t+1}}{b_t'(\widehat{n}_t) - \frac{u_3'(\widehat{x}_t)}{u_1'(\widehat{x}_t)}} = \frac{u_1'(\widehat{x}_t)}{u_2'(\widehat{x}_t)} = D_1 F(\widehat{k}^o_{t+1}, \widehat{n}_t).$$

Observe that $W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})$ is the maximum utility that an individual born at $t-1$, endowed with $\widehat{e}_t$ units of physical resources, can obtain if they are constrained to provide each agent of the next generation with $\widehat{e}_{t+1}$ units of resources. Notice also that by monotonicity of preferences, each function W_{t-1} must be strictly increasing on e_t and strictly decreasing on e_{t+1}. By Proposition 1, the question of whether an allocation $\widehat{a}$ satisfying the necessary conditions in (4) can be reduced to a issue on whether or not there exists a sequence $\{\widetilde{e}_t\}_{t=0}^{\infty}$ such that $W_{t-1}(\widetilde{e}_t, \widetilde{e}_{t+1}) \geq W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})$ for all $t \geq 0$, and $W_{\tau-1}(\widetilde{e}_\tau, \widetilde{e}_{\tau+1}) > W_{\tau-1}(\widehat{e}_\tau, \widehat{e}_{\tau+1})$ for some $\tau \geq 0$.

Observe that an allocation satisfying the necessary condition in Proposition 1 cannot be improved upon by a reallocation of resources involving a finite number of generations. However, this condition does not guarantee that such an allocation is efficient, as a reallocation of resources involving an infinite sequence of intergenerational transfers might improved on it. This is stated formally in the following Lemma, closely related to Balasko and Shell (1980, Lemma 5.4).

Lemma 1 *Let $\widehat{a} \in \mathcal{S}^*$ be an allocation satisfying the necessary conditions in Proposition (1), and suppose $\widehat{a}$ is $\mathcal{M}-$inefficient. Then there exists an allocation $\widetilde{a} \in \mathcal{S}^*$ that Millian dominates the allocation $\widehat{a}$, and some period $T \geq 0$, such that,*

$$\widetilde{e}_t \leq \widehat{e}_t \text{ for all } t \geq 0, \text{ and } e_\tau < \widehat{e}_\tau \text{ for all } \tau \geq T. \tag{9}$$

Proof. Let $\widehat{a} \in \mathcal{S}^*$ be an inefficient allocation satisfying the conditions (5)–(8), and let $\widetilde{a}$ be an allocation that Millian dominates the allocation $\widehat{a}$, that is, an allocation satisfying

$$W_{t-1}(\widetilde{e}_t, \widetilde{e}_{t+1}) \geq W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1}) \text{ and } W_{T-1}(\widetilde{e}_T, \widetilde{e}_{T+1}) > W_{T-1}(\widehat{e}_T, \widehat{e}_{T+1}) \text{ for some } t = T.$$

To show $\widetilde{a}$ satisfies condition (9), observe first that $\widetilde{a}$ verifies $\widetilde{e}_0 \leq \widehat{e}_0$, where the inequality must be strict if $U_{-2}(\widetilde{a}) > U_{-2}(\widehat{a})$. Taking into account that $W_{-1}(\cdot)$ is strictly increasing in e_0 one obtains

$$W_{-1}(\widetilde{e}_0, \widehat{e}_1) \leq W_{-1}(\widehat{e}_0, \widehat{e}_1).$$

Also, since $W_{-1}(\cdot)$ is strictly decreasing in e_1, the inequality $W_{-1}(\widetilde{e}_0, \widetilde{e}_1) \geq W_{-1}(\widehat{e}_0, \widehat{e}_1)$ is only satisfied if $\widetilde{e}_1 \leq \widehat{e}_1$, where the last inequality must be strict if either $W_{-1}(\widetilde{e}_0, \widetilde{e}_1) > W_{-1}(\widehat{e}_0, \widehat{e}_1)$ or $\widetilde{e}_0 < \widehat{e}_0$ is satisfied. Proceeding analogously, since $W_0(\cdot)$ is strictly decreasing in $\widehat{e}_2$ and the inequality $W_0(\widetilde{e}_1, \widetilde{e}_2) \geq W_0(\widehat{e}_1, \widehat{e}_2)$ must be satisfied one must have $\widetilde{e}_2 \leq \widehat{e}_2$ (with $\widetilde{e}_2 < \widehat{e}_2$ if either $W_0(\widetilde{e}_1, \widetilde{e}_2) > W_0(\widehat{e}_1, \widehat{e}_2)$ or $\widetilde{e}_1 < \widehat{e}_1$ holds). By applying the argument recursively one obtains

$$\widehat{e}_t - \widetilde{e}_t \geq 0 \text{ for all } t \geq 0$$

and

$$\widehat{e}_\tau - \widetilde{e}_\tau > 0 \text{ for some } T \text{ and all } \tau \geq T,$$

which establishes condition (9) and, therefore, completes the proof of Lemma 1.□

In view of Lemma 1, we will adopt the terminology proposed by Balasko and Shell (1980) and refer to an allocation satisfying the necessary conditions in Proposition 1 as a *statically* (or short-run) $\mathcal{M}-$efficient allocation.

Remark 1. Recall that given a sequence $\{\lambda_\tau\}_{\tau \geq -2}$ of intergenerational weights, a Millian social welfare function $\mathcal{V}_\lambda : \mathcal{S} \longrightarrow \Re$ is a function defined, for every $a \in \mathcal{S}$, by $\mathcal{V}_\lambda(a) = \sum_{t=-1}^{\infty} \lambda_{t-1} U_{t-1}(a)$. It is straightforward to show that any interior allocation $\widehat{a}$ maximizing a Millian welfare function among those feasible symmetric allocations must satisfy the necessary conditions (5)–(8). However, even if $\mathcal{V}_\lambda(\widehat{a}) < \infty$ is satisfied for an allocation $\widehat{a}$ verifying the first order conditions (5)–(8), it is not in general true that such an allocation maximizes $\mathcal{V}_\lambda$ on the set of feasible allocations $\mathcal{S}$. As we mentioned above and will become clear along the paper, the set $\mathcal{S}$ is non-convex, and therefore first order conditions might be not sufficient for a maximum. For this reason, the previously mentioned result stating that a Millian optimum must be a Millian efficient allocation is not particularly useful, since it says nothing on whether or not an allocation verifying the first order conditions of a Millian optimization problem is indeed a Millian optimum.

3.2 Sufficient conditions. Dynamic $\mathcal{M}-$efficiency.

In the previous subsection we have obtained necessary conditions for Millian efficiency, but those conditions do not guarantee that an allocation solving the sequence of optimization problems in the statement of Proposition 1 is actually $\mathcal{M}-$efficient. Well known theoretical work deals with the issue of efficiency in dynamic models, such as Cass (1972) in the context of a simple physical capital growth model, and Balasko and Shell (1980) who focus on an overlapping generations exchange economy. These papers show that, despite an allocation can be *short-run* efficient (or statically efficient), i.e., it cannot be improved upon by a reallocation of resources of a finite number of generations, it might not be *long-run* efficient (or dynamically efficient), that is, fully efficient. In this subsection, we extend these previous results to an environment of endogenous population.

Recall from the previous section that if preferences and technologies are represented by differentiable functions, then an interior solution $(x_t(\widehat{e}_t, \widehat{e}_{t+1}), k_{t+1}(\widehat{e}_t, \widehat{e}_{t+1}) >> 0$ to the optimization problem in the definition of $W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})$ is characterized by its first order conditions (5)–(8). Since W_{t-1} is strictly monotonic, the indifference curve given by all pairs (e_t, e_{t+1}) such that $W_{t-1}(e_t, e_{t+1}) = W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})$ implicitly defines e_{t+1} as a continuous, strictly increasing function of e_t. Define $m_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})$ as the slope of this indifference curve at the point $(\widehat{e}_t, \widehat{e}_{t+1})$, which is always well defined. The Implicit Function Theorem yields

$$m_{t-1}(\widehat{e}_t, \widehat{e}_{t+1}) = -\frac{\frac{\partial W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})}{\partial e_t}}{\frac{\partial W_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})}{\partial e_{t+1}}} = -\frac{\lambda_t(\widehat{e}_t, \widehat{e}_{t+1})}{\lambda_{t+1}(\widehat{e}_t, \widehat{e}_{t+1})},$$

where $\lambda_t(\widehat{e}_t, \widehat{e}_{t+1})$ and $\lambda_{t+1}(\widehat{e}_t, \widehat{e}_{t+1})$ are the Kuhn-Tucker multipliers for which the first order conditions of the optimization problem (4) are satisfied. Taking this into account we obtain, by the Envelope Theorem,

$$\lambda_t(\widehat{e}_t, \widehat{e}_{t+1}) = u_1'(x_t(\widehat{e}_t, \widehat{e}_{t+1})),$$

and

$$\lambda_{t+1}(\widehat{e}_t, \widehat{e}_{t+1}) = -u_2'(x_t(\widehat{e}_t, \widehat{e}_{t+1}))n_t(\widehat{e}_t, \widehat{e}_{t+1}),$$

which by letting $R_{t+1}(\widehat{e}_t, \widehat{e}_{t+1}) = u_1'(x_t(\widehat{e}_t, \widehat{e}_{t+1}))/u_2'(x_t(\widehat{e}_t, \widehat{e}_{t+1}))$ yields

$$m_{t-1}(\widehat{e}_t, \widehat{e}_{t+1}) = \frac{R_{t+1}(\widehat{e}_t, \widehat{e}_{t+1})}{n_t(\widehat{e}_t, \widehat{e}_{t+1})}. \tag{10}$$

In an exogenous population setting, the ratio $m_t(\widehat{e}_t, \widehat{e}_{t+1})$ defined above is crucial to determine whether or not a an allocation is dynamically efficient. As we show below, things are slightly different when the rate of population growth is endogenous.

3.2.1 Dynamic efficiency in the stationary case. We will first gain some intuitions by considering stationary allocations (that is, allocations such that $a_t = a_{t+1}$ for all $t \geq 0$) in a stationary economy (i.e., such that $F_t \equiv F$ and $b_t(n_t) \equiv b(n_t)$ for all $t \geq 0$). Note that in this case one has $W_{t-1} \equiv W$ and $m_{t-1} \equiv m$ for all $t \geq 0$, that is, the indirect utility function and the function determining the slope of the indifference curve passing through any point are the same for all generations. Since for any such stationary allocations one has $e_t = e_{t+1} = e$ for all $t \geq 0$, the set of stationary allocations is represented by the line $e_t = e_{t+1}$ in $\Re^2_+$.

To simplify things, assume first that W is strictly quasiconcave, that is, the slope of any indifference curve (which is given by $m(e_t, e_{t+1}) = R(e_t, e_{t+1})/n(e_t, e_{t+1})$) decreases as e_t increases. Consider now a point like (e', e') in Figure 1, corresponding to a stationary allocation a' satisfying the necessary conditions (5)–(8), with $e'_t = e'_{t+1} = e'$. Note that for such allocation one has $m(e', e') < 1$. Clearly, such allocation a' is not efficient since by reducing e' towards the point e^g in the figure, all agents are better off. By contrast, consider now a point like (e'', e''), corresponding to an allocation a'' for which $e''_t = e''_{t+1} = e''$ and $m(e'', e'') > 1$. Apparently, it is possible to improve all agents by increasing e'' in the direction of e^g. However, achieving Pareto improvements by increasing e'' is impossible, because increasing e'' in period $t = \tau$ implies that agents born at time $t = \tau - 2$ (i.e., the old generation at period τ) necessarily decrease their consumption and, hence, their utility. Thus, such allocation a'' cannot be dominated by any other stationary allocation.

To summarize, it is straightforward to prove that if W is quasiconcave, then a necessary and sufficient condition ensuring that a stationary allocation $\widehat{a}$ is not dominated by any other stationary allocation is

$$m(\widehat{e}, \widehat{e}) \geq 1.$$

In what follows, we will refer to this condition as the Phelps-Koopmans-Diamond [PKD] condition (see Galor and Ryder, 1991). Note that if W is quasiconcave, condition PKD can be written equivalently as $e'' \leq e^g$, where e^g is the endowment corresponding to a stationary allocation a^g for which $m(e^g, e^g) = 1$. Such allocation a^g has been referred to as the golden rule allocation, since it maximizes the utility obtained by a representative agent among those feasible stationary allocations.

In the case that the indirect utility function is not quasiconcave, condition $m(e, e) = R(e, e)/n(e, e) < 1$ is still valid to conclude that a statically efficient stationary allocation $\widehat{a}$ is not dynamically efficient. However, condition $m(e, e) = R(e, e)/n(e, e) \geq 1$ no longer guarantees that a statically efficient stationary allocation a is also dynamically efficient. This case is illustrated in Figure 2, in which a point $(\widetilde{e}, \widetilde{e})$ corresponding to an allocation verifying condition PKD does not correspond to an efficient allocation. Observe that the line with slope $m(\widetilde{e}, \widetilde{e})$ passing through $(\widetilde{e}, \widetilde{e})$ does not separate the upper contour set of this point. The flattest line passing through the point $(\widetilde{e}, \widetilde{e})$

separating the upper contour set of $(\widetilde{e},\widetilde{e})$ is given by the dotted line in the figure. We will use the slope of this line to provide an alternative criterium ensuring that an allocation is efficient.

Given a pair $(\widehat{e}_t,\widehat{e}_{t+1})$, define

$$\pi_{t-1}(\widehat{e}_t,\widehat{e}_{t+1}) = \inf_{(e_t,e_{t+1})<<(\widehat{e}_t,\widehat{e}_{t+1})} \left\{ \frac{\widehat{e}_{t+1}-e_{t+1}}{\widehat{e}_t-e_t} : W_{t-1}(e_t,e_{t+1}) \geq W_{t-1}(\widehat{e}_t,\widehat{e}_{t+1}) \right\} \quad (11)$$

That is, $\pi_{t-1}(\widehat{e}_t,\widehat{e}_{t+1})$ is the flattest line that can be drawn by joining $(\widehat{e}_t,\widehat{e}_{t+1})$ with a pair $(e_t,e_{t+1}) << (\widehat{e}_t,\widehat{e}_{t+1})$ in the upper contour set of $(\widehat{e}_t,\widehat{e}_{t+1})$. Notice that

$$\pi_{t-1}(\widehat{e}_t,\widehat{e}_{t+1}) \leq \frac{\widehat{R}_{t+1}(\widehat{e}_t,\widehat{e}_{t+1})}{\widehat{n}_t(\widehat{e}_t,\widehat{e}_{t+1})} = m_{t-1}(\widehat{e}_t,\widehat{e}_{t+1}),$$

where the inequality above holds as a strict equality whenever W_{t-1} is quasiconcave.

With this notation, the set of stationary allocations that are not Millian dominated by any other stationary allocation is characterized as follows.

Proposition 2 *(i) Let $\widehat{a} \in \mathcal{S}^*$ be a stationary allocation (of a stationary economy) such that $\widehat{e}_t = \widehat{e}_{t+1} = \widehat{e}$ for all $t \geq 1$. Then $\widehat{a}$ is not $\mathcal{M}-$dominated by any other stationary allocation if and only if*

$$\pi_{t-1}(\widehat{e},\widehat{e}) = \pi(\widehat{e},\widehat{e}) \geq 1. \quad (12)$$

(ii) Let $\widehat{a} \in \mathcal{S}^*$ be a stationary allocation for which $\widehat{e}_t = \widehat{e}$ for all t, and suppose that $m(e,e) > 1$ for all $0 \leq e \leq \widehat{e}$. Then, the stationary allocation $\widehat{a}$ verifies $\pi_t(\widehat{e},\widehat{e}) = \pi(\widehat{e},\widehat{e}) > 1$.

Proof. The proof of (i) is straightforward from the definition of π_{t-1} and Lemma 1. In order to prove (ii) first observe that, given that W_{t-1} is increasing in e_t and decreasing in e_{t+1}, any stationary point below $(\widehat{e},\widehat{e})$ belongs to an indifference curve that provides strictly lower welfare; that is, given that $(\widehat{e},\widehat{e})$ belongs to an indifference curve $\mathcal{I}_{\widehat{w}}$, then any $(\widetilde{e},\widetilde{e})$ with $\widetilde{e} < \widehat{e}$ verifies $\widetilde{e} \in \mathcal{I}_{\widetilde{w}}$, with $\widetilde{w} < \widehat{w}$. Consequently, the slope of the line defined above passing through $(\widehat{e},\widehat{e})$ will be strictly larger than 1: $\pi(\widehat{e},\widehat{e}) > 1$.$\square$

To conclude the analysis of the stationary case, we illustrate our results through three examples. In Example 1 we identify the stationary Millian allocations in an environment with a Cobb-Douglas production function. In Examples 2 and 3, with a CES and a linear production function, respectively, we show that the standard PKD condition may fail to identify efficient allocations.

Example 2. A Cobb-Douglas production function. Let $u(x_t) = (c_t^m)^{1-\delta-\gamma}\left(c_{t+1}^o\right)^{\delta}(n_t)^{\gamma}$ with $\delta+\gamma<1$, and δ, $\gamma \in (0,1)$; $F(k_t^o,n_{t-1}) = A\,(k_t^o)^{\alpha}(n_{t-1})^{1-\alpha}$ with $\alpha \in (0,1)$ and $A>0$; and $b_t(n_t) = \beta n_t$. To simplify the algebra we will write the necessary conditions (5)–(8) in terms of capital per worker, $k_t^m = k_t^o / n_{t-1}$. Let $f(k_t^m) = F(k_t^m, 1) = A\,(k_t^m)^{\alpha}$. In this economy, $x(e_t, e_{t+1})$ is given by

$$c^m(e_t, e_{t+1}) = (1 - \delta - \gamma)\, e_t; \qquad c^o(e_t, e_{t+1}) = \delta e_t f'(k^m(e_{t+1})); \;\; \text{and,}$$
$$n(e_t, e_{t+1}) = (\delta + \gamma)\left(\frac{e_t}{\beta + k^m(e_{t+1})}\right)$$

where $k^m(\cdot)$ is an increasing function of e_{t+1} implicitly defined, for all $e_{t+1} \geq 0$, by

$$f(k^m(e_{t+1})) - \left(\frac{\delta}{\delta + \gamma}\right) f'(k^m(e_{t+1}))\,[\beta + k^m(e_{t+1})] = e_{t+1}.$$

By substituting in the definition of $k^m(e)$ we get

$$\begin{aligned} z(e) = m(e,e) &= \frac{R(e,e)}{n(e,e)} = \frac{1}{(\delta+\gamma)} \frac{(\beta + k^m(e))\, f'(k^m(e))}{e} = \\ &= \left[(\delta + \gamma)\frac{f(k^m(e))}{f'(k^m(e))\,[\beta + k^m(e)]} - \delta\right]^{-1}. \end{aligned}$$

Thus, the behavior of $m(e,e)$ depends of the behavior of the function $H(k^m) = \frac{f(k^m)}{f'(k^m)[\beta + k^m]}$, which for this case it is given by $H(k^m) = \frac{k^m}{\alpha[\beta + k^m]}$ and $H'(k^m) = \frac{\beta}{\alpha[\beta + k^m]^2} > 0$. Since $k^m(\cdot)$ is a strictly increasing function satisfying $k^m(0) = 0$, the function $z(\cdot)$ must also be decreasing on $\Re_{++}$ and satisfies $z(0) > 0$. In this case the PKD condition $m(\widehat{e}, \widehat{e}) \geq 1$ (i.e., $z(\widehat{e}) \geq 1$) correctly identifies efficient allocations, since for every $\widehat{e}$ such that $m(\widehat{e}, \widehat{e}) \geq 1$ one must have $m(\widehat{e}, \widehat{e}) > 1$ for every $\widehat{e} < e$, which in turn by Proposition 2.(ii) yields $\pi(\widehat{e}, \widehat{e}) \geq 1$. Finally notice that if $\left(\frac{\delta + \gamma}{1 - \delta}\right) < \alpha$, then $m(\widehat{e}, \widehat{e}) > 1$ for all $e \geq 0$, which implies that every stationary allocation is dynamically efficient (or, to be more precise, no stationary allocation can be dominated by any other stationary allocation).□

Example 3. A CES production function. Consider the same economy as that in Example 2 but with a CES production function $F(k_t^o, n_{t-1}) = [A\,(k_t^o)^{\rho} + B\,(n_{t-1})^{\rho}]^{1/\rho}$ with $\rho \leq 1$ and $A, B > 0$. Let $f(k_t^m) = F(k_t^m, 1) = [A\,(k_t^m)^{\rho} + B]^{\frac{1}{\rho}}$. The functions $H()$ and $m()$ are defined as in Example 2. For this case, the function H is given by

$$H(k^m) = \frac{[A\,(k^m)^{\rho} + B]^{\frac{1}{\rho}}}{A\,(k^m)^{\rho - 1}\,[(k^m)^{\rho} + B]^{\frac{1}{\rho} - 1}\,[\beta + k^m]} = \frac{k^m}{\beta + k^m} + \frac{B\,(k^m)^{1-\rho}}{A\,[\beta + k^m]},$$

and therefore the equation $m(e,e) = 1$ can be written equivalently as

$$(\delta + \gamma)\left(Ak^m(e) + B\,(k^m(e))^{1-\rho}\right) = (1 + \delta)A\,[\beta + k^m(e)]$$

Observe that, in this case, the functions H and m are not increasingly monotonic, and therefore equation $m(e,e) = 1$ might have two positive solutions. For the particular case in which $\delta = 1/3$, $\gamma = 1/3$, $A = 1$, $B = 3$, $\rho = 1/2$, and $\beta = 1$ equation $z(e) = 1$ can be written as

$$0 = k^m(e) + 2 - 3\,(k^m(e))^{\frac{1}{2}},$$

and it has two positive roots at $\underline{e} = 12$ and $\overline{e} = 18.75$, such that $k^m(\underline{e}) = 1$ and $k^m(\overline{e}) = 4$ respectively, and one has $m(\underline{e}, \underline{e}) = m(\overline{e}, \overline{e}) = 1$. We show in Figure 3 that the PKD criterium might fail to identify efficient allocations. Figure 3.a displays the value of the $R(e,e)/n(e,e)$ ratio for each of the stationary allocations, and indicates those stationary allocations $e \in [0, 12] \bigcup [18.75, +\infty)$ that fulfill the PKD criterium, i.e., $m(e,e) \geq 1$. Figure 3.b represents the welfare obtained at each stationary allocation, $W(e,e)$, and identifies that not all of these stationary allocations are efficient. Since W achieves a local maximum at $\underline{e} = 12$ and there exists a stationary amount of resources $e^* = 24.84$ such that $W(\underline{e}, \underline{e}) = W(e^*, e^*)$, it is easy to identify that the region of (dynamically) efficient allocations stands for $e \in [0, 12] \bigcup [24.84, +\infty)$, while the allocations that fall into the region $e \in (12, 24.84)$ are dynamically inefficient. The latter region includes those allocations verifying the PKD condition, but they are not dynamically efficient, in particular $e \in [18.75, 24.84)$.□

Example 4. A linear production function. Consider the same economy as that in Example 2 but with a linear production function $F_t(k_t^o, n_{t-1}) = Rk_t^o + \omega n_{t-1}$, where R, $\omega > 0$. Let $f(k_t^m) = F(k_t^m, 1) = Rk_t^m + \omega$. Observe that here the optimization problem (4) in the definition of W requires additionally the non-negativity constraint for capital accumulation. Consequently, we had to distinguish between two cases, whether the capital per worker is positive or zero. First, the equation that defines implicitly the function $k^m()$ in Example 2 sets a lower threshold, such that if $e_{t+1} \geq \omega - \frac{\delta}{\delta+\gamma}\beta R \geq 0$ the capital per worker is strictly positive, and then $x(e_t, e_{t+1})$ is found as indicated there. However, whenever $e_{t+1} \in \left[0, \omega - \frac{\delta}{\delta+\gamma}\beta R\right)$ is satisfied, the non-negativity constraint on capital is binding, i.e., $k^m(e_{t+1}) = 0$, and one has $c_t^m(e_t, e_{t+1}) = (1 - \delta - \gamma)e_t$, $c_{t+1}^o(e_t, e_{t+1}) = n_t(e_t, e_{t+1})(\omega - e_{t+1})$, and $n_t(e_t, e_{t+1}) = ((\delta + \gamma)/\beta)\, e_t$. Therefore, the indirect utility function W adopts the form

$$W_{t-1}(e_t, e_{t+1}) = \begin{cases} \Theta_1 \left[\omega - e_{t+1}\right]^{\delta} e_t & \text{if } e_{t+1} \in \left[0, \omega - \frac{\delta}{\delta+\gamma}\beta R\right); \\ \Theta_2 \left[\beta R + e_{t+1} - \omega\right]^{-\gamma} e_t & \text{if } e_{t+1} \geq \omega - \frac{\delta}{\delta+\gamma}\beta R \geq 0 \end{cases}$$

where Θ_1 and Θ_2 are constants depending on the parameters. It follows that W_{t-1} is strictly quasiconcave on the set $\left\{(e_t, e_{t+1}) : 0 \leq e_{t+1} < \omega - \frac{\delta}{\delta+\gamma}\beta R\right\}$, and strictly quasiconvex on the set $\left\{(e_t, e_{t+1}) : e_{t+1} \geq \omega - \frac{\delta}{\delta+\gamma}\beta R\right\}$.

To conclude the example, consider the stationary allocation a^* for which $e^* = \frac{w-\beta R}{1-\gamma}$ verifies $m(e^*, e^*) = 1$; that is, a^* would be identified as a "dynamically efficient" allocation according to the PKD criterium. Suppose that $\frac{w-\beta R}{1-\gamma} \geq w - \frac{\delta}{\delta+\gamma}\beta R \geq 0$ is satisfied. Since W is strictly quasiconvex around e^*, this allocation must be a local minimum and, therefore, any stationary allocation a' such that for a sufficient close $e' < e^*$, $\mathcal{M}-$dominates the allocation a^*.□

3.2.2 A sufficient condition for dynamic efficiency of non-stationary paths. Although non-convexities arising in the model make it difficult to extend the analysis of stationary paths to provide a complete characterization of (non necessarily stationary) dynamic efficient paths, the following proposition provides a sufficient condition that guarantees that an allocation is efficient even if the indirect utility function W_{t-1} fails to be quasiconcave.

PROPOSITION 3 *Consider an allocation $\widehat{a} \in \mathcal{S}^*$ satisfying the necessary condition in Proposition 1. If*

$$\lim_{T\to\infty} \inf \left(\frac{\widehat{e}_T}{\prod_{t=0}^{T} \pi_t \left(\widehat{e}_t, \widehat{e}_{t+1}\right)} \right) = 0, \tag{13}$$

then $\widehat{a}$ is Millian efficient.

Proof. Consider an allocation $\widehat{a} \in \mathcal{S}$ satisfying conditions (5)–(8), and (13), and suppose now that it is not efficient. To show that this yields a contradiction, let $\widetilde{a}$ be an allocation dominating the allocation $\widehat{a}$, and let τ be the first period for which $W_{\tau-1}(\widetilde{e}_\tau, \widetilde{e}_{\tau+1}) > W_{\tau-1}(\widehat{e}_\tau, \widehat{e}_{\tau+1})$. Observe that by Lemma 1, $\widetilde{e}_\tau < \widehat{e}_\tau$ must be satisfied, and therefore there exists ϵ_τ such that $\epsilon_\tau = \widehat{e}_\tau - \widetilde{e}_\tau > 0$. Since $\widehat{a}$ satisfies condition (13), there must exist a sufficiently large T^* such that, for each $T > T^*$ one has

$$\left(\frac{\widehat{e}_T}{\prod_{t=0}^{T} \pi_t \left(\widehat{e}_t, \widehat{e}_{t+1}\right)} \right) < \epsilon_\tau = \widehat{e}_\tau - \widetilde{e}_\tau.$$

Use now condition (9) in the statement of Lemma 1 and the definition of $\pi_t \left(\widehat{e}_t, \widehat{e}_{t+1}\right)$ to obtain the chain of inequalities

$$\begin{aligned} 0 \;&<\; (\widehat{e}_\tau - \widetilde{e}_\tau) = \epsilon_\tau \leq \frac{(\widehat{e}_{\tau+1} - \widetilde{e}_{\tau+1})}{\pi_\tau \left(\widehat{e}_\tau, \widehat{e}_{\tau+1}\right)} \leq \frac{\widehat{e}_{\tau+2} - \widetilde{e}_{\tau+2}}{\pi_\tau \left(\widehat{e}_\tau, \widehat{e}_{\tau+1}\right) \pi_{\tau+1} \left(\widehat{e}_{\tau+1}, \widehat{e}_{\tau+2}\right)} \leq \\ &\leq\; \dots \leq \\ &\leq\; \frac{\widehat{e}_T - \widetilde{e}_T}{\prod_{t=\tau}^{T} \pi_t \left(\widehat{e}_t, \widehat{e}_{t+1}\right)} < \frac{\widehat{e}_T}{\prod_{t=\tau}^{T} \pi_t \left(\widehat{e}_t, \widehat{e}_{t+1}\right)}, \end{aligned}$$

which contradicts condition (13) and, therefore, establishes that $\widehat{a}$ is Millian efficient.□

4 A CHARACTERIZATION OF $\mathcal{M}-$EFFICIENT ALLOCATIONS AS DECENTRALIZED EQUILIBRIA

In this section, we provide an alternative characterization of interior statically efficient allocations as the equilibria of a decentralized price mechanism with intergenerational transfers.

This price mechanism is described as follows. There are two markets operating at each date $t \geq 0$: a financial market, that allows agents to lend (or borrow) an arbitrary amount s_t of the homogeneous good in period t, and obtain (or pay back) a return equal to R_{t+1} units of the same good in period $t+1$; and, a spot job market, in which labor is exchanged against the homogeneous good at a price w_t. In addition, there exists an intergenerational transfer program, represented by a sequence $T = \{T_t\}_{t \geq 0}$, that collects a lump sum tax T_t from every middle-aged adult living at t, and gives every old adult a transfer P_t that depends linearly on the number of children that he has, according to the rule $P_t = n_{t-1}T_t$. Note that such intergenerational policy is balanced every period.

With these two markets operating at each date, the life-cycle optimization problem for an agent born in period $t-1$, with $t = 0, 1, ...$, is

$$
\begin{aligned}
V_{t-1}(w_t - T_t, R_{t+1}, T_{t+1}) \quad = \quad & \max_{(x_t, s_t)} u(c_t^m, c_{t+1}^o, n_t) \\
& \text{subject to: } c_t^m + s_t + b_t(n_t) \leq w_t - T_t \\
& c_{t+1}^o \leq R_{t+1}s_t + n_t T_{t+1};
\end{aligned} \tag{14}
$$

given R_{t+1}, w_t, T_t and T_{t+1}; and where s_t represents the middle-aged agent's savings at period t.

Labor and physical capital are purchased by firms at competitive prices equal, respectively, to $w_t = D_2F_t(k_t^o, n_{t-1})$ and $R_t = D_1F_t(k_t^o, n_{t-1})$; and, aggregate savings finances investment in physical capital, $s_t = k_{t+1}^o$. For the initial condition $(n_{-2}, n_{-1}, k_0^o) = \left(1, \overline{n}_{-1}, \overline{K}_0\right)$, and a given *sequence of contracts* $T = \{T_t\}_{\tau=0}^{\infty}$, a *decentralized equilibrium* (generated by T) is an allocation a^* and a sequence of prices $\{w_\tau^*, R_\tau^*\}_{\tau=0}^{\infty}$, which are characterized by

$$\frac{u_1'(x_t^*)}{u_2'(x_t^*)} = R_{t+1}^*; \tag{15}$$

$$\left[b'(n_t^*) - \frac{u_3'(x_t^*)}{u_1'(x_t^*)}\right] \frac{u_1'(x_t^*)}{u_2'(x_t^*)} = T_{t+1}; \tag{16}$$

$$w_{t+1}^* = D_1F_{t+1}(k_{t+1}^{o*}, n_t^*); \tag{17}$$

$$R_{t+1}^* = D_2F_{t+1}(k_{t+1}^{o*}, n_t^*); \tag{18}$$

$$s_t^* = k_{t+1}^{o*}; \tag{19}$$

and, the two individual budget constraints in (14) for all $t \geq 0$.

Observe that the second condition holds for any fees pattern, including $T_{t+1} = 0$. These equations provide a straightforward characterization of statically $\mathcal{M}-$efficient interior allocations as the equilibria of the sequential price mechanism described above, as the following result states.

Theorem 4

(i) Consider an arbitrary sequence T^* of intergenerational contracts, and let a^* be an interior decentralized equilibrium generated by T^*. Then a^* is statically $\mathcal{M}-$efficient.

(ii) For every statically $\mathcal{M}-$efficient interior allocation a^*, there exists a sequence T^* of intergenerational contracts generating a^* as a decentralized equilibrium.

Proof. To prove (i), let $(a^*, \{w_t^*, R_t^*\}_{t\geq 0})$ be an interior decentralized equilibrium generated by a sequence T, and denote $e_t = w_t^* - T_t$ as this equilibrium net resources owned by a middle-aged agent born at period $t-1$. First, it is straightforward to check that a^* is a feasible allocation satisfying (5)–(7) for each $t \geq 0$. Then, in order to show that a^* satisfies also (8), we make use the budget constraint for the old adult agent in (14), the condition (19), and the fact that at competitive equilibrium prices maximum profits are equal to zero at all periods.

To prove (ii), let a^* be an interior statically $\mathcal{M}-$efficient allocation verifying the necessary conditions (5)–(8). Then, it is straightforward to check that a sequence of intergenerational transfers $\{T_{t+1}\}_{t=-1}^{\infty}$ defined by (16) generates a^* as an interior decentralized equilibrium allocation.□

Thus, the notion of Millian efficiency admits a characterization that is closely analogous to the one provided by the two fundamental Theorems of Welfare Economics. Nevertheless, two important differences arise.

With respect to the statement (i) in Theorem 4, that can be regarded as a version of the First Fundamental Welfare Theorem, it is important to observe that the equilibrium generated without an intergenerational transfer program $T_t = 0$ for all t, is statically $\mathcal{M}-$efficient; that is, there is no need to subsidize children on (static) efficiency grounds. Despite Groezen, Leers and Medjam (2003) have arrived to similar conclusions in the context of a small open economy with Cobb-Douglas utility functions, they regard as efficient any allocation satisfying the first order conditions of a Millian welfare maximization problem, which might be incorrect. In fact, their parametric small open economy is essentially equivalent to the economy with linear technology and exogenous wages and interest rates that we study in our Example 4 above, which yields indirect utility functions that are not quasiconcave.

The statement (ii) of Theorem 4 can be regarded as a version of the Second Fundamental Theorem of Welfare Economics. Similar to the case of economies with exogenous population, every Millian efficient allocation can be decentralized by initially selecting an appropriate sequence of intergenerational transfers, and then allowing the agents to determine their consumption and investment decisions at competitive markets. Differently from the standard, exogenous population case, in which non-distorting intergenerational transfers must be lump-sum for all agents, an incentive scheme that links intergenerational transfers with fertility decisions is needed. More precisely, for every system of intergenerational transfers that achieves Millian efficiency, every middle-aged adult has to pay a tax $T_t > 0$ (or, in some cases, receive a lump-sum subsidy given by $T_t < 0$), while every old adult has to receive a subsidy (or pay a tax) which depends linearly of the number of children she decides to have.

4.1 Dynamic efficiency and competitive prices.

The characterization given above refers to statically $\mathcal{M}-$efficient allocations. Of course, many of those equilibria described above might be dynamically inefficient, and therefore it is worth exploring which of these equilibria ensure that dynamic efficiency is achieved. One way to proceed consists in computing the sequence $\left\{\pi_t(e_t^*, e_{t+1}^*)\right\}_{t\geq 0}$ associated to an equilibrium path and then apply Proposition 3. However, computing each $\pi_t(e_t^*, e_{t+1}^*)$ may involve some difficulties since it requires computing $W_{t-1}(e_t, e_{t+1})$ for every pair $(e_t, e_{t+1}) \leq (e_t^*, e_{t+1}^*)$. The Proposition stated below provides a simpler condition for dynamic efficiency that exclusively uses the sequence of prices $p^* = \{w_t^*, R_t^*\}_{t=0}^{\infty}$ associated to any decentralized equilibrium. In order to prove this result, we need to introduce the following technical assumption, which implies that an individual is able to achieve any utility level by having enough children.

Assumption 1 *For every $\overline{u}$ in the range of u, and every $(c_t^m, c_{t+1}^o) \in \Re_+^2$, there exists a n_t such that $u(c_t^m, c_{t+1}^o, n_t) \geq \overline{u}$.*

Proposition 5 *Assume that the utility function u satisfies Assumption 1, and let (a^*, p^*) be a decentralized equilibrium generated by a sequence T^* of intergenerational transfers. Then a^* is Millian efficient if*

$$\lim_{T\to\infty} \inf \left(\frac{e_T^*}{\prod_{t=0}^{T} \frac{e_{t+1}^* + \left(\beta_t R_{t+1}^* - w_{t+1}^*\right)}{e_t^*}} \right) = 0$$

is satisfied, where $e_t^* = w_t^* - T_t^*$ and $\beta_t = b_t'(0)$ for all t.

Proof. By Lemma 7, which is formally proved in the technical appendix, for every t and every $(\widehat{e}_t, \widehat{e}_{t+1})$, $\pi_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})$, defined as in (11), is bounded by

$$\pi_{t-1}(e_t, e_{t+1}) \geq \frac{e_{t+1}^* + \left(\beta_t R_{t+1}^* - w_{t+1}^*\right)}{e_t^*},$$

and then, the proof is completed by applying the sufficient condition in Proposition 3.□

Finally, an interesting result for stationary economies, straightforward proved from the previous proposition, is the following;

Proposition 6 *An interior decentralized equilibrium a^* (generated by T^*) that converges to a steady state with constant competitive equilibrium prices (w^*, R^*) is Millian efficient if*

$$R^* - \frac{w^*}{b'(0)} > 0, .$$

Consequently, the PKD criterium defined for the case of exogenous fertility must be replaced by this alternative criterium, that states that the rate of return to physical capital should be highest than a term that corresponds to the highest rate of return (in terms of goods) to invest in children, i.e., $w^*/b'(0)$.

5 Conclusions

This paper is concerned with Pareto efficiency in an overlapping generations setting with endogenous fertility decisions. In an environment in which the set of agents is endogenous, we explore the properties of an extension of the notion of Pareto efficiency, referred to as *Millian efficiency.* The notion of Pareto dominance underlying the Millian notion is based exclusively on the preference profiles of those agents alive, and allows only to ranking symmetric allocations with positive fertility rates through every period. We show that when fertility decisions are endogenous, the set of feasible allocations in overlapping generations economies is non-convex, and the standard sufficient conditions for dynamic efficiency in an exogenous population environment cannot be straightforward applied. In the paper, we provide necessary conditions that every Millian efficient allocation must verify, and a sufficient condition determining whether a given allocation that satisfies these necessary conditions is Millian efficient.

With these results at hand, we adapt the Fundamental Theorems of Welfare Economics to a setting with endogenous population by characterizing every (statically) Millian efficient allocation as the equilibrium of a decentralized sequential price mechanism. Similar to the case of economies with exogenous population, every Millian efficient allocation can be decentralized by initially selecting an appropriate sequence of intergenerational transfers, and then allowing the agents to determine their consumption and investment decisions at competitive markets. Differently from the standard exogenous population case, an incentive scheme that links intergenerational transfers with fertility decisions is needed. More precisely, for every system of intergenerational transfers that achieves Millian efficiency, every middle-aged adult has to pay a tax (or, in some cases, receive a subsidy), and every old adult will receive a subsidy (or pay a tax) which depends linearly of the number of children she decided to have. As a particular case, we also show that the allocation corresponding to a decentralized equilibrium with no intergenerational transfers, for which there is no need to subsidize or tax children, is (statically) Millian efficient.

To summarize, the theoretical study on the notion of efficiency with endogenous populations presented in this paper has relevant implications for analyzing the role of social security programs in achieving optimal intergenerational redistribution. First, empirical tests of dynamic efficiency based on the PKD criterium might be no longer valid. Second, optimal intergenerational trade might be reached by spontaneous agreement of the agents involved. Third, if a government wishes to enforce intergenerational transfers, a mechanism linking these transfers with fertility decisions is needed.

Several extensions, such as allowing for more general forms of altruism between the agents and their descendants or introducing human capital accumulation by children, would be worth exploring. These two extensions would provide a more general setting to discuss any proposal on the role that some institutions, such as the family or the institutions comprising the welfare state, should play in achieving optimal intergenerational trade. Some authors have pointed out that, in addition to the market failure caused by dynamic inefficiencies, other types of market failures might affect intergenerational trade. For example, Becker and Murphy (1988), or Boldrin and Montes (2005) have argued that children might not have access to capital markets to finance their human capital accumulation; and, Rangel (2003) has argued that the elderly cannot rely on the markets to obtain some goods. If the optimal rate of return to investment in children is affected by intergenerational transfers (as it occurs in the setting studied in this paper) any public policy that enforces intergenerational transfers on efficiency grounds should take into account the effect of intergenerational transfers on fertility choices.

References

[1] Balasko, Yves; and Karl Shell (1980) "The Overlapping-Generations Model I: The Case of Pure Exchange without Money," *Journal of Economic Theory* 23, pp.281-306.

[2] Becker, Gary S.; and Kevin M. Murphy (1988) "The Family and the State," *Journal of Law and Economics* 31 (1), pp.1-18.

[3] Bental, B. (1989) "The Old Age Security Hypothesis and Optimal Population Growth," *Journal of Population Economics* 1, pp.285-301.

[4] Boldrin, Michele; and, Ana Montes (2005) "The Intergenerational welfare state. Public Education and Pensions," *Review of Economic Studies* vol.73, n.3.

[5] Cass, David (1972) "On Capital Overaccumulation in the Aggregative, Neoclassical Model of Economic Growth: A Complete Characterization," *Journal of Economic Theory* 4, pp.200-223.

[6] Cigno, Alessandro (1992) "Children and Pensions," *Journal of Population Economics* 5, pp.175-83

[7] Cigno, Alessandro (1993) "Intergenerational Transfers without Altruism. Family, Market and State," *European Journal of Political Economy* 9, pp.505-518.

[8] Cigno, Alessandro (2003) "The political economy of intergenerational cooperation," in S.C. Kolm and J. Mercier Ythier (eds.), *Handbook of the Economics of Giving, Reciprocity and Altruism*, Amsterdam: North-Holland, *forthcoming*

[9] Conde-Ruiz, José Ignacio; Eduardo L. Giménez; and Mikel Pérez-Nievas (2004) "Millian Efficiency with Endogenous Fertility," *Documento de Trabajo* 2004-13. FEDEA, Madrid.

[10] Deardorff, Alan V. (1976) "The Optimum Growth Rate for Population: Comment," *International Economic Review* 17, n.2 pp.510-515.

[11] Diamond, Peter A. (1965) "National Debt in a Neoclassical Growht Model," *American Economic Review* 55, pp.1126-1150.

[12] Eckstein, Z.; and K. I. Wolpin (1985) "Endogenous Fertility and Optimal Population Size," *Journal of Public Economics* 27, pp.93-106.

[13] Galor, Oded; and Harl E. Ryder (1991) "Dynamic efficiency of steady-state equilibria in an overlapping-generations model with productive capital," *Economics Letters* 35, pp.365-390.

[14] Groezen, Bas van; Theo Leers; and Lex Meijdam (2003) "Social Security and Endogenous Fertility: Pensions and Child Allowances as Siamese Twins," *Journal of Public Economics* 87, pp.233-251.

[15] Golosov, Mikhail; Larry E. Jones; and Michèle Tertilt (2004) "Efficiency with Endogenous Population Growth," NBER *Working Paper* w10231.

[16] Koopmans, T.C. (1963) "On the Concept of Optimal Economic Growth," Cowles Foundation *Discussion Paper* n.163.

[17] Michel, Philippe; and P. Pestieau (1993) "Population Growth and Optimality: When Does Serendipity hold", *Journal of Population Economics*, pp.353-362

[18] Michel, Philippe; and Bertrand Wigniolle (2003) "On Efficient Child Making,"*mimeo*, EUREQua, Paris.

[19] Nerlove, Marc L.; Assaf Razin; and Efraim Sadka (1982) "Population Size and the Social Welfare Functions of Bentham and Mill," *Economic Letters* 10, pp.61-4.

[20] Nerlove, Marc L.; Assaf Razin; and Efraim Sadka (1985) "Population Size: Individual Choice and Social Optima," *Quarterly Journal of Economics* 100, pp.321-34.

[21] Phelps, Edmund S. (1965) "Second Essay on the Golden Rule of Accumulation," *American Economic Review* LV, n.4, pp.783-814.

[22] Rangel, Antonio (2003) "Backward and Forward Intergenerational goods: Why is Social Security Good for the Environment?,"*American Economic Review*, 93, pp.813-34.

[23] Raut, Lakshmi Kanta (1992) "Effect of Social Security on Fertility and Savings: An Overlapping Generations Model," *Indian Economic Review* vol XXVII, No 1, pp.25-43.

[24] Razin, Assaf; and Efrain Sadka (1995) *Population Economics*, Cambridge, The MIT Press.

[25] Samuelson, Paul A. (1975) "The Optimum Growth Rate for Population," *International Economic Review* 16, n.3 pp.531-538.

[26] Samuelson, Paul A. (1976) "The Optimum Growth Rate for Population: Agreement and Evaluations," *International Economic Review* 17, n.2 pp.516-525.

Lemma 7 *For every t and every $(\widehat{e}_t, \widehat{e}_{t+1})$, $\pi_{t-1}(\widehat{e}_t, \widehat{e}_{t+1})$, defined as in (11), is bounded by*

$$\pi_{t-1}(e_t, e_{t+1}) \geq \frac{e^*_{t+1} + \left(\beta_t R^*_{t+1} - w^*_{t+1}\right)}{e^*_t}. \tag{20}$$

Proof. We begin by defining, for an arbitrary period t and a given pair (w^*_t, R^*_{t+1}), the function

$$\begin{aligned} W^*_{t-1}(e_t, e_{t+1}; w^*_t, R^*_{t+1}) &= V_{t-1}(e_t, R^*_{t+1}, w^*_{t+1} - e_{t+1}) \\ &= \max_{(x_t, k^o_{t+1}) \in \Re^3_+ \times \Re} \Big\{ u(x_t) : c^m_t + \beta_t n_t + k^o_{t+1} = e_t; \\ &\qquad c^o_{t+1} = R^*_{t+1} k^o_{t+1} + w^*_{t+1} n_t - n_t e_{t+1} \Big\}. \end{aligned} \tag{21}$$

To prove (20) we proceed by steps.

Step 1. We first prove that, for every $(e_t, e_{t+1}) \leq \left(e^*_t, e^*_{t+1}\right)$, we have $W^*_{t-1}(e_t, e_{t+1}; w^*_t, R^*_{t+1}) \geq W_{t-1}(e_t, e_{t+1})$, where $W_{t-1}(e_t, e_{t+1})$ is defined as in (4).

To prove Step 1, let $(e_t, e_{t+1}) \leq \left(e^*_t, e^*_{t+1}\right)$ be arbitrary and let $x_t = x_t(e_t, e_{t+1})$ and $k^o_{t+1} = k^o_{t+1}(e_t, e_{t+1})$ be a solution to the optimization problem in the definition of $W_{t-1}(e_t, e_{t+1})$. Since, at competitive equilibrium prices, maximum profits are achieved at (k^{o*}_{t+1}, n^*_t) and are equal to zero, we have

$$F_{t+1}(k^{o*}_{t+1}, n^*_t) - R^*_{t+1} k^{o*}_{t+1} - w^*_{t+1} n^*_t = 0 \geq F_{t+1}(k^o_{t+1}, n_t) - R^*_{t+1} k^o_{t+1} - w^*_{t+} n_t.$$

Therefore, the pair (x_t, k_{t+1}) satisfies

$$c^o_{t+1} \leq F_{t+1}(k^o_{t+1}, n_t) - n_t e_{t+1} \leq R^*_t k_{t+1} + w^*_{t-1} n_{t-1} - n_t e_{t+1}$$

and, since b_t is a convex function satisfying $b_t(n_t) \geq \beta_t n_t$, we know that $c^m_t + \beta_t n_t + k^o_{t+1} \leq e_t$. Thus, the pair (x_t, k_{t+1}) is also feasible in the optimization problem in the definition of W^*_{t-1}, a contradiction that establishes that $W^*_{t-1}(e_t, e_{t+1}; w^*_t, R^*_{t+1}) \geq W_{t-1}(e_t, e_{t+1})$ must be satisfied.

Step 2. We now show that $\pi_{t-1}(e^*_t, e^*_{t+1}) \geq \pi^*_{t-1}(e^*_t, e^*_{t+1}; w^*_t, R^*_{t+1})$, where π^*_{t-1} is defined as

$$\begin{aligned} &\pi^*_{t-1}(e^*_t, e^*_{t+1}; w^*_t, R^*_{t+1}) = \\ &= \inf_{(e_t, e_{t+1}) << \left(e^*_t, e^*_{t+1}\right)} \left\{ \frac{e^*_{t+1} - e_{t+1}}{e^*_t - e_t} : W^*_{t-1}(e_t, e_{t+1}; w^*_t, R^*_{t+1}) \geq W^*_{t-1}(e^*_t, e^*_{t+1}; w^*_t, R^*_{t+1}) \right\}. \end{aligned}$$

To prove Step 2, first note that $W^*_{t-1}(e^*_t, e^*_{t+1}; w^*_t, R^*_{t+1}) = W_{t-1}(e^*_t, e^*_{t+1})$ is satisfied from Theorem 4.i). This fact implies that the indifference curves $I(e^*_t, e^*_{t+1})$ and $I^*(e^*_t, e^*_{t+1})$ defined as

$$\begin{aligned} I(e^*_t, e^*_{t+1}) &\equiv \left\{ (e_t, e_{t+1}) \geq 0 : W_{t-1}(e_t, e_{t+1}) = W_{t-1}(e^*_t, e^*_{t+1}) \right\} \\ I^*(e^*_t, e^*_{t+1}) &\equiv \left\{ (e_t, e_{t+1}) \geq 0 : W^*_{t-1}(e_t, e_{t+1}; w^*_t, R^*_{t+1}) = W^*_{t-1}(e^*_t, e^*_{t+1}; w^*_t, R^*_{t+1}) \right\} \end{aligned} \tag{22}$$

coincide at (e_t^*, e_{t+1}^*); and, by Step 1, the indifference curve $I(e_t^*, e_{t+1}^*)$ lies below $I^*(e_t^*, e_{t+1}^*)$ for any pair $(e_t, e_{t+1}) \leq (e_t^*, e_{t+1}^*)$. Since for any pair $(e_t, e_{t+1}) \leq (e_t^*, e_{t+1}^*)$, $\pi^*(e_t^*, e_{t+1}^*; w_t^*, R_{t+1}^*)$ and $\pi(e_t^*, e_{t+1}^*)$ are the slopes of the flattest lines above $I^*(e_t^*, e_{t+1}^*)$ and $I(e_t^*, e_{t+1}^*)$ respectively, this establishes Step 2.

Step 3. It only remains to be shown that

$$\pi_{t-1}^*(e_t^*, e_{t+1}^*; w_t^*, R_{t+1}^*) \quad = \quad \frac{e_{t+1}^* + \left(\beta_t R_{t+1}^* - w_{t+1}^*\right)}{e_t^*} \tag{23}$$

is satisfied. To prove this statement, we will specify the properties of the indifference curve $I^*(e_t^*, e_{t+1}^*)$, namely it is convex and it crosses the $e_{t+1}-$axis at the point $\beta_t R_{t+1}^* - w_t^*$.

First define the vector of prices $\mathbf{p}_t^* = (p_{mt}^*, p_{nt}^*(e_{t+1}), p_{ot}^*) = (1, \beta_t - (w_{t+1}^* - e_{t+1})/R_{t+1}^*, 1/R_t^*)$ for all t. Then the maximization problem (21) can be equivalently written at the standard consumer problem

$$W_{t-1}^*(e_t, e_{t+1}; w_t^*, R_{t+1}^*) \quad = \quad \max_{x_t \in \Re_+^3 \geq 0} \left\{u(x_t) : c_t^m + p_{nt}^* n_t + p_{ot}^* c_{t+1}^o = e_t\right\},$$

whose corresponding expenditure minimization problem is

$$E(\mathbf{p}_t; \overline{u}) = \min_{x_t \in \Re_+^3 \geq 0} \left\{p_{mt} c_t^m + p_{nt} n_t + p_{ot} c_{t+1}^o : u(x_t) \geq \overline{u}\right\}.$$

Since the utility function is non-decreasing and satisfies Assumption 1, the expenditure function $E(\mathbf{p}_t; \overline{u})$ verifies the standard properties: it is non-decreasing in $\mathbf{p}_t$, concave and continuous in $\mathbf{p}_t$, and satisfies $\lim_{p_n \to 0} E(1, p_c, p_n; \overline{u}) = 0$. Notice also that for the given level of $\overline{u} = W_{t-1}^*(e_t^*, e_{t+1}^*; w_t^*, R_{t+1}^*)$, the expenditure function $E(\mathbf{p}_t^*; W_{t-1}^*(e_t^*, e_{t+1}^*; w_t^*, R_{t+1}^*)) = e_t$ defines implicitly the indifference curve $I^*(e_t^*, e_{t+1}^*)$ in (22). Concretely, the implicit relationship between e_t and e_{t+1} is

$$E((1, p_{nt}^*(e_{t+1}), p_{ot}^*); W_{t-1}^*(e_t^*, e_{t+1}^*; w_t^*, R_{t+1}^*)) - e_t = 0.$$

Applying the Implicit Function Theorem to the indifference curve defined implicitly in the previous equation, it is easy to see that $I^*(e_t^*, e_{t+1}^*)$ is convex and increasing.

Finally, in order to prove that the indifference curve crosses the $e_{t+1}-$axis at the point $\beta_t R_{t+1}^* - w_t^*$, notice that at any period t the unique way to achieve a positive level of welfare $\overline{u} = W_{t-1}^*(e_t^*, e_{t+1}^*; w_t^*, R_{t+1}^*) > 0$ is for $p_{nt}^*(e_{t+1}) = \beta_t - (w_{t+1}^* - e_{t+1})/R_{t+1}^* = 0$.

Consequently, (23) defines $\pi^*(e_t^*, e_{t+1}^*; w_t^*, R_{t+1}^*)$, i.e., the flattest slope that pass through the pair (e_t^*, e_{t+1}^*) and leaves below the indifference curve $I^*(e_t^*, e_{t+1}^*)$ for $(e_t, e_{t+1}) \leq (e_t^*, e_{t+1}^*)$. This concludes Step 3.

The proof is complete by making use Steps 2 and 3.□

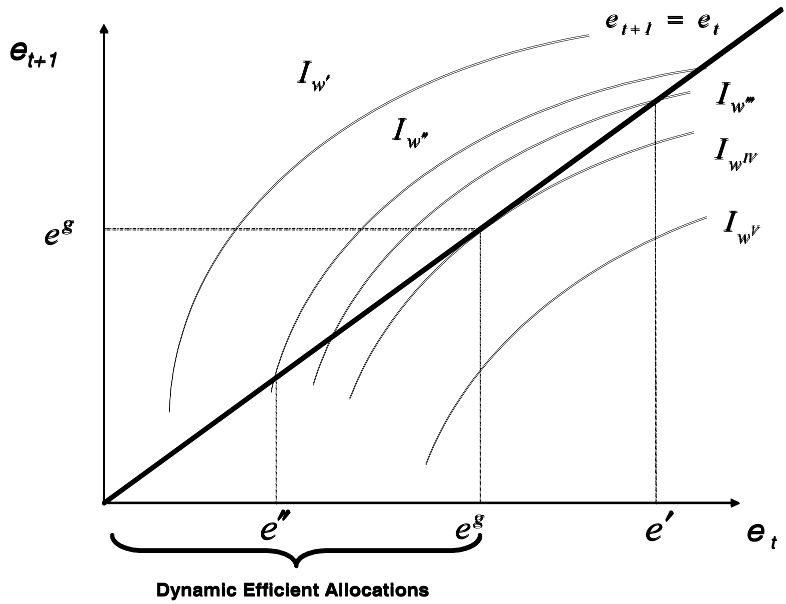

Figure 1: Stationary Dynamic efficient allocations when W is strictly quasiconcave.

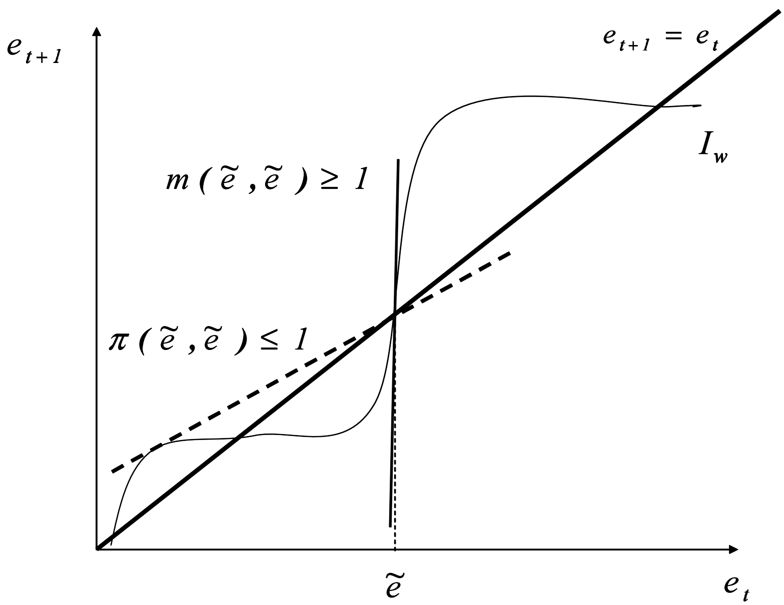

Figure 2: An example when W is not strictly quasiconcave of a stationary allocation $\tilde{e}$ that verifies the P-D-K condition, but is not efficient .

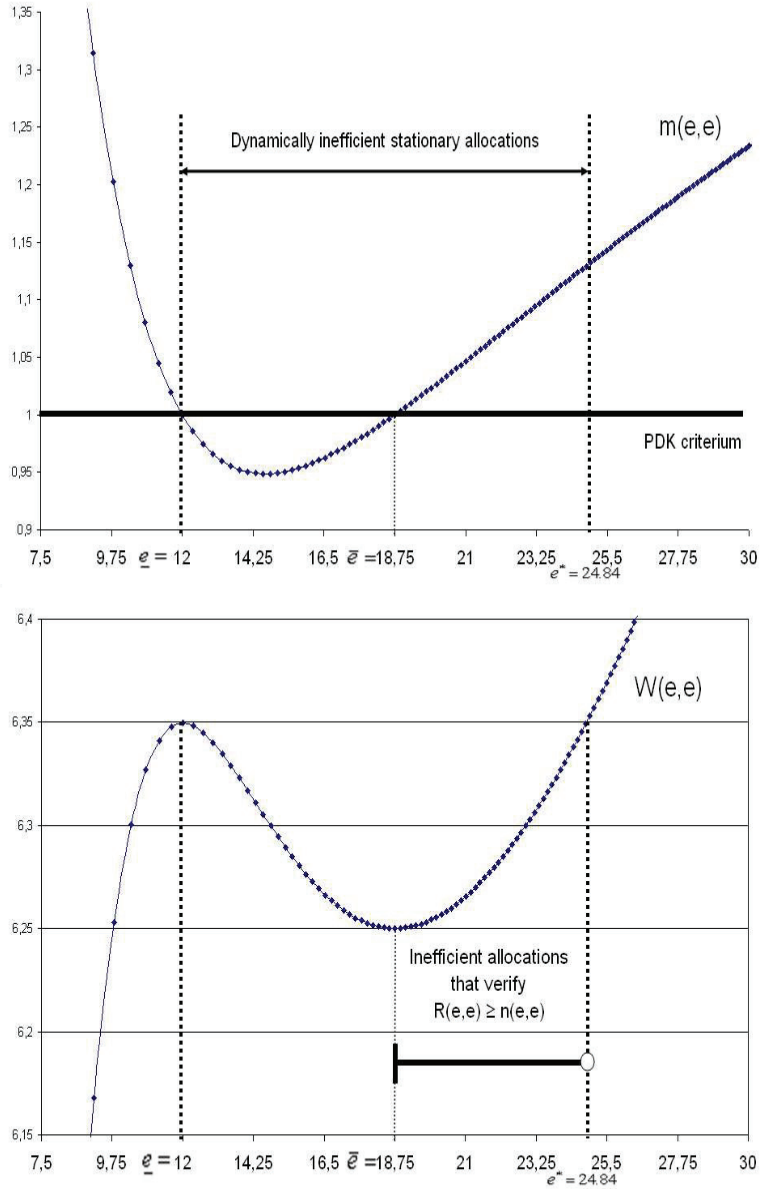

Figure 3: Numerical example where the PKD criterium fails to identify dynamically efficient allocations. Above the $R(e,e)/n(e,e)$ function. Below the indirect utility function for the stationary allocations. (Parameters $\delta = 1/3$, $\gamma = 1/3$, $A = 1$, $B = 3$, $\rho = 1/2$, and $\beta = 1$.)

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