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XOSÉ LUIS QUIÑOÁ LÓPEZ
Universidade de Santiago de Compostela

ABOUT THE SIZE OF THE MEASUREMENT UNITS IN A LEONTIEF SYSTEM AND ITS CONSEQUENCES

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About the size of the measurement units in a Leontief system and its consequences

Xosé Luis Quiñoá López[1]

ABSTRACT

Beginning from the economic meaning of indecomposability in an Leontief-type system and then modifying the size of the measure units of the different goods through consecutive approximations, the original system is turned into another, structurally equivalent, one whose technological matrix A is such that $\forall j, \quad \sum_i a_{ij} = \overline{a}$, the maximum eigenvalue of A that admits $\overline{1} = (1,...,1)$ as left positive eigenvector, from where we deduce the main Perron-Frobenius theorem.

Keywords: Leontief system, Units size, Consecutive approximations, Perron-Frobenius.

[1] University of Santiago de Compostela, Faculty of Economic and Business Sciences, Department of Quantitative Economy, Northern University Campus – 15782 Santiago de Compostela
Phone Number.: 34 881 8 11516 – joseluis.quinoa@usc.es

INTRODUCTION

A Leontief system is represented though a board

$$\left[\left(q_{ij}\right) = \begin{pmatrix} q_{11} & \cdots & q_{1j} & \cdots & q_{1n} \\ \cdots & & q_{ij} & \cdots & q_{in} \\ q_{n1} & \cdots & q_{nj} & \cdots & q_{nn} \end{pmatrix}; \ \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_n \end{pmatrix}; \ Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_i \\ \vdots \\ Q_n \end{pmatrix} \right] \qquad (1)$$

in which:

- (q_{ij}) represents the interindustrial transactions matrix; each term q_{ij} shows the physical quantity of the goods i used by the industry j in the considered period of time.
- $\beta = (\beta_i)$ is the column vector that represents the surplus of the system; β_i is the surplus of the industry i, and we suppose that it exists at least one industry in which $\beta_i > 0$.
- $Q = (Q_i)$ is the column vector that represents the total output of the different goods in the considered period.
- $L = (L_1, \ldots, L_i, \ldots, L_n)$ is the row vector of the labour quantities used by the different industries.
- $A = (a_{ij})$ shows the square matrix $n \times n$ defined by $a_{ij} = \dfrac{q_{ij}}{Q_j}$, and that we will name the system´s technological matrix.
- $l = (l_i)$ is the row vector defined by $l_i = \dfrac{L_i}{Q_i}$; l_i represents the amount of labour used in the production of a product´s unit i.
- We will denote as $\begin{bmatrix} A \\ l \end{bmatrix}$ what we know as the "system´s technique".

Each column matrix (q_{ij}) or (a_{ij}) represents heterogeneous goods, whose respective quantities depend on the measurement units (kilograms, tons, dozens...). From that, we deduce that the system type (1) can be represented by a countless matrix (q_{ij}) or (a_{ij}), we only need to modify the size of one of the measurement units for obtaining a different matrix.

Our target is to find a technological matrix *A* that could be considered as "characteristic" in a given Leontief system.

We will see that this is like taking the size of one the measurement units as numeraire, and according to it, the other units can be expressed. This will be possible for indecomposable systems, for certain decomposable systems, and even for some infinite ones.

Let´s consider the example suggested by Sraffa at the beginning of *Production of Commodities by Means of Commodities*, relating to two iron and wheat industries, whose quantities had been measured respectively in tons and quarters. In Leontief terminology, we represent the problem as

$$\left[(q_{ij}) = \begin{pmatrix} 280 & 120 \\ 12 & 8 \end{pmatrix} \; ; \; \beta = \begin{pmatrix} 175 \\ 0 \end{pmatrix} \; ; \; Q = \begin{pmatrix} 575 \\ 20 \end{pmatrix}\right] \qquad (2)$$

with a technological matrix

$$A \approx \begin{pmatrix} 0{,}486956 & 6 \\ 0{,}020869 & 0{,}4 \end{pmatrix}$$

The technological matrix *A* is

$$S_1 = \sum_i a_{i1} = 0{,}507825 \quad ; \quad S_2 = \sum_i a_{i2} = 6{,}4$$

What suggests that the quarter is a measurement unit too small relative to the ton, making the total numerical quantity of wheat, Q_1 = 575, excessively large in comparison with the iron $Q_2 = 20$.

Let´s suppose that we take 5 quarters for wheat and $\frac{1}{3}$ tons for iron as measurement units. The system results:

$$\left[(q'_{ij}) = \begin{pmatrix} 56 & 24 \\ 36 & 24 \end{pmatrix} \; ; \; \beta' = \begin{pmatrix} 35 \\ 0 \end{pmatrix} \; ; \; Q' = \begin{pmatrix} 115 \\ 60 \end{pmatrix}\right] \qquad (3)$$

whose technological matrix is:

$$A' \cong \begin{pmatrix} 0{,}486956 & 0{,}4 \\ 0{,}313043 & 0{,}4 \end{pmatrix} = \left(a'_{ij}\right)$$

which has the important property that

$$S'_1 = \sum_i a'_{i1} = 0{,}8 \quad ; \quad S'_2 = \sum_i a'_{i2} = 0{,}8$$

The system (3) is structurally equivalent to the original (2), with the only difference that the same physical quantity of the goods is being expressed in different measurement units, giving as a as result different numerical quantities.

We can see that:

a) $\bar{a} = 0{,}8$ is the maximum eigenvalue of A (and A').

b) $\left(\frac{1}{5}, 3\right)$ (or any possible multiple) of it is the positive eigenvector on the left associated to the maximum eigenvalue of A, $\bar{a} = 0{,}8$. Likewise, $\bar{1} = (1,...,1)$ is the positive eigenvector on the left of A' associated to $\bar{a} = 0{,}8$ (the wheat units are five times bigger and the iron ones three times smaller).

c) The Leontief inverse of A' is:

$$(I - A')^{-1} \cong \begin{pmatrix} 3{,}2857 & 2{,}1904 \\ 1{,}7142 & 2{,}8095 \end{pmatrix} = (\alpha_{ij})$$

and it has the property that

$$\sum_i \alpha_{i1} \cong 5 = \frac{1}{1-\bar{a}} \quad ; \quad \sum_i \alpha_{i2} \cong 5 = \frac{1}{1-\bar{a}}$$

d) In the system (3) the ratio between means of production used $\sum_{i,j} q'_{ij}$ and total production $\sum Q'_i$ results

$$\frac{\sum_{i,j} q'_{ij}}{\sum Q'_i} = \frac{140}{175} = 0{,}8 = \overline{a}$$

and we also deduce the ratios according to $\overline{a}$ between surplus and total output, and surplus and used production means.

e) Taking $r = (r_1, r_2) = \left(\frac{1}{5}, 3\right)$ as eigenvector on the left of A associated to $\overline{a}$ =0,8, and $\hat{r} = \begin{pmatrix} 1/5 & 0 \\ 0 & 3 \end{pmatrix}$ we have $A' = \hat{r}\, A\, \hat{r}^{-1}$.

In the example, as we saw, we take as unit 5 quarters for wheat and $\frac{1}{3}$ tons for iron, which is the same as saying that in the system 1 ton of iron is equal to 15 quarters of wheat, and that is the accurate conclusion reached by Sraffa: "*The exchange ratio which the advances to be replaced and the profits to be distributed to both industries in proportion to their advances is 15 qr. of wheat for 1 t. of iron, and the corresponding rate of profits in each industry is 25%*".

Regrettably, Sraffa –who seems to have sensed the importance of considering the size of the measurement units- did not continue on that way, maybe because his efforts were straightly directed to the distribution problem.

If matrix A is indecomposable and the system has any kind of surplus, then the theorem of Perron-Frobenius indicates that its maximum eigenvalue is $\overline{a} < 1$, for which it corresponds an eigenvector on the left $r = (r_1, ..., r_i, ..., r_n)$, such that $\forall i, \quad r_i > 0$.

Then, the converted system

$$\left[(r_i\, q_{ij}),\ (r_i\, \beta_i),\ r_i\, Q_i \right] \qquad (4)$$

Is such that its technological matrix A'

$$A' = \hat{r}\, A\, \hat{r}^{-1} = (a'_{ij})$$

has the property that

$$\forall j, \qquad \sum_i a'_{ij} = \overline{a}$$

And we demonstrate that we have all the properties a), b), c), d) and e) of the example of Sraffa.

What we want to do is to follow the inverse way, to define the economical meaning of the indecomposability with any size of the measurement units and, then, by modifying their size through consecutive approximations, to convert the original system into another structurally equivalent one, whose technological matrix A' is such that

$$\forall j, \qquad \sum_i a'_{ij} = \overline{a}$$

maximum eigenvalue of A' (and A), that admits $\overline{1} = (1,...,1)$ as associated positive eigenvector on the left, and from where the Perron-Frobenius theorem is deduced.

DECOMPOSABILITY AND GRAPH THEORY

We will say that the system (1) is decomposable –or reducible– if it exists a subset $\{i_1,\ldots, i_p\}$, $1 \leq p < n$ of the system´s industries, such that these industries don´t use products of the other industries. By negation we will say that the system is decomposable –or irreducible– if any non-empty subset of industries –any industry in particular *i*– uses directly or indirectly products of the other industries.

A system is mathematically decomposable if its technological matrix A –or q_{ij}– can be converted through row and column changes to the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ \theta & A_{22} \end{pmatrix}$$

in which A_{11} and A_{22} are square sub-matrices, being θ a null sub-matrix.

In the analysis that we propose it is very useful to interpret decomposability through elemental graph theory.

DEFINITION. If $N = \{1,\ldots, j,\ldots, n\}$ is the set of industries in the system, we will denominate as *graph* the whole application Γ of N in the set $P(N)$ of parts of N. The elements of N are the *vertex* of the graph.

In a Leontief system with its technological matrix $A = (a_{ij})$, we define the graph as:

$$\Gamma(i) = \{ j \in N \mid a_{ij} \neq 0 \} \subset N$$

We will call arc of the graph Γ to every pair (i, j) of vertex (industries), such that $j \in \Gamma_{(i)}$; which is the same as saying that the industry j directly uses the product i.

We will say that i is the initial vertex and j the final one, simply denoting as $i \to j$.

We can represent the N elements as points in the plane, in which case we describe Γ through the set of its arcs.

For example, given the matrix

$$A = \begin{pmatrix} 0{,}3 & 0 & 0{,}2 \\ 0{,}1 & 0 & 0{,}4 \\ 0{,}2 & 0{,}6 & 0{,}1 \end{pmatrix} \quad \begin{array}{ll} \Gamma(1) = \{1, 3\}, & 1 \to 1,\ 1 \to 3 \\ \Gamma(2) = \{1, 3\}, & 2 \to 1,\ 2 \to 3 \\ \Gamma(3) = \{1, 2, 3\}, & 3 \to 1,\ 3 \to 2,\ 3 \to 3 \end{array}$$

We will call *way* to every series $i_1, i_2,\ldots, i_k$ of vertex, such that $i_{j+1} \in \Gamma(i_j)$, i.e.,

$$i_1 \to i_2 \to \ldots \to i_{k-1} \to i_k$$

where the vertex i_1 is the beginning of the way, and vertex i_k is the end.

We will say that the industry j *is accessible* from i if it exists a way:

$$i = i_1 \to i_2 \to \ldots \to i_{k-1} \to i_k = j$$

If $a_{ij} \neq 0$, then $i \to j$, which means that industry j directly uses the product i, while if $a_{ij} = 0$ and j is accessible from i, the industry $j = i_k$ directly uses the product $i_{k-1} \ldots$ that uses i_2, that uses the $i_1 = i$; i.e. industry j indirectly uses product i.

We will say that the graph Γ of the system *is strongly connected* if for all industries i and j there is a beginning of the way i and an ending of the way j, meaning that industry j uses product i, directly if $a_{ij} \neq 0$, or indirectly if $a_{ij} = 0$.

We will denominate *circuit* C_i to every way whose beginning i is confused with its end in i: $i = i_1 \to i_2 \to \ldots \to i_k = i$, and we will say that C_i *is complete* if

$$\{i_1, i_2, \ldots, i_k\} = \{1, \ldots, j \ldots, n\} = N$$

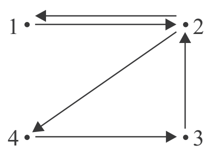
i.e. that starting from a vertex i it is possible to get "back" to i by a way that goes through all vertex of the graph.

From previous conditions, we can give the following interpretation of the decomposability of the system.

If it exists a complete circuit $c\,(i)$ for all i, it means that each industry uses the products of the other industries directly or indirectly, and that the system is indecomposable.

EXAMPLE. For the matrix

$$A = \begin{pmatrix} 0 & 0{,}8 & 0 & 0 \\ 0{,}5 & 0 & 0 & 1 \\ 0 & 0{,}2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



the correspondent complete circuits are:

$$1 \to 2 \to 4 \to 3 \to 2 \to 1$$

$$2 \to 4 \to 3 \to 2 \to 1 \to 2$$

$$3 \to 2 \to 1 \to 2 \to 4 \to 3$$

$$4 \to 3 \to 2 \to 1 \to 2 \to 4$$

For each vertex i there is a complete circuit, so that A is indecomposable.

In general, each column j of A shows heterogeneous quantities of the goods *directly* used in the production of an unit of good j, but the goods indirectly used and their numerical quantities don´t appear.

In the previous example, industry 1 only uses the product of 2 which uses the products of 1 and 3, using this last product the one of 4; that is to say, that industry 1 uses products of the other industries direct or indirectly in its production, and we can say the same about the industries 2, 3 and 4.

Let´s consider again the Leontief system that we described in (1) and a $r = (r_1,\dots, r_i,\dots, r_n) \in \mathrm{R}^n$, such that $\forall i,\quad r_i > 0$, and modify the size of each unit i according to r_i. The system is converted into:

$$\left(q'_{ij}\right) = \begin{pmatrix} r_1 q_{11} & \cdots & r_1 q_{1j} & \cdots & r_1 q_{1n} \\ & \cdots & \cdots & \cdots & \\ r_i q_{i1} & \cdots & r_i q_{ij} & \cdots & r_i q_{in} \\ & \cdots & \cdots & \cdots & \\ r_n q_{n1} & \cdots & r_n q_{nj} & \cdots & r_n q_{nn} \end{pmatrix}; \quad \beta' = \begin{pmatrix} r_1 \beta_1 \\ \vdots \\ r_i \beta_i \\ \vdots \\ r_n \beta_n \end{pmatrix}; \quad Q' = \begin{pmatrix} r_1 Q_1 \\ \vdots \\ r_i Q_i \\ \vdots \\ r_n Q_n \end{pmatrix} \qquad (5)$$

The system (5) is structurally equivalent to (1), as each $q'_{ij} = r_i\, q_{ij}$ represents the same physical quantity of output i numerically expressed in different size units.

The technological matrix of the new system, $A' = \left(a'_{ij}\right)$ is:

$$a'_{ij} = \frac{r_i\, q_{ij}}{r_j\, Q_j} = \frac{r_i}{r_j} a_{ij}$$

And denoting as $\hat{r}$ the diagonal matrix whose diagonal elements are the r_i, we have $A' = \hat{r}\, A\, \hat{r}^{-1}$, deducing that the characteristical equations of A and A'

$$\det(\lambda I - A') = \det(\lambda I - \hat{r}\, A\, \hat{r}^{-1}) = \det\left[\hat{r}\,(\lambda I - A)\,\hat{r}^{-1}\right] = \\ = \det \hat{r} \,.\, \det(\lambda I - A) \,.\, \det \hat{r}^{-1} = \det(\lambda I - A)$$

are the same, and thus that A and A' have the same eigenvalues.

BANACH ALGEBRA M_n (R) AND PERRON-FROBENIUS THEOREM

The vectorial space M_n (R) of the square matrix of order n with the rule

$$\|A\| = \sup_j \sum_i |a_{ij}|$$

or, alternatively,

$$\|A\|^x = \sup_i \sum_j |a_{ij}|$$

has Banach algebraic structure with unit I, being the rule compatible with the matrix product[2]:

$$\|A.B\| \leq \|A\| \,.\, \|B\|$$

$$\|I\| = 1$$

Being A the technological matrix of a Leontief system and for each j, $1 \leq j \leq n$, let´s say

$$S_j = \sum_i |a_{ij}|$$

[2] See mathematical appendix.

We have, then

$$\|A\| = \sup_j \{S_j\}$$

If all the S_j are equal to $\|A\| = \overline{a}$, then $\overline{a}$ is the maximum eigenvalue of A that admits $\overline{1} = (1,...,1,...,1)$ as eigenvector on the left, and $\forall \lambda \in$ R, $\lambda > \|A\|$, we have that $(\lambda I - A)$ is inversible and that $(\lambda I - A)^{-1} \geq 0$[3].

Given a technological matrix A with rule $a = \|A\|$, and considering the matrix $\frac{A}{a}$, with rule 1. We will follow the reasoning with a matrix of rule ≤ 1

It is well known that if $\|A\| < 1$, $I - A$ is inversible and

$$(I - A)^{-1} = I + A + A^2 + \ldots + A^n + \ldots$$

The problem of the inversibility of $I - A$ appears when $\|A\| = 1$.

Now, we propose a version of the Perron-Frobenius theorem with a demonstration based on topological considerations and the economic interpretation of the unit´s size modification through consecutive approximations.

THEOREM. Being A a technological matrix such that

$$\|A\| = \sup_j \sum a_{ij} = 1$$

if A is indecomposable, and at least for one j,

$$S_j = \sum_i a_{ij} < 1$$

then $I - A$ is inversible, its inverse is positive, and there exists $r \in$ R^n, $r > 0$, such that the matrix $A' = \hat{r}\, A\, \hat{r}^{-1}$ verify that $\forall j$,

$$S'_j = \sum_i a'_{ij} = \overline{a}$$

[3] See Mathematical appendix.

is a maximum eigenvalue of A, strictly minor than 1, and that admits $\overline{1} = (1,...,1)$ as associated eigenvector on the left.

DEMONSTRATION. Let´s demonstrate, first, that it exists $r \in \mathbb{R}^n$, $r > 0$, such that $\|\hat{r} A \hat{r}^{-1}\| < 1$.

Given $r_i \in \mathbb{R}$, $r_i > 0$, we will denote $\overline{r_i}$ the vector $\mathbb{R}^n$ whose components are all equal to 1, except the one of order i, which is equal to r_i.

Being i an industry such that

$$S_i = \sum_k a_{ki} < 1$$

If for every $j \neq i$ $a_{ij} \neq 0$, it means that all industries j directly use the product i, so it is enough to take r_i, $S_i < r_i < 1$ and then, the addition of the elements of the columns $j \neq i$ of $\hat{\bar{r}} A \hat{\bar{r}}^{-1}$ will be:

$$S'_j = a_{1j} + ... + a_{ij} r_i + ... + a_{nj} < a_{1j} + ... + a_{ij} + a_{nj} = S_j \leq 1$$

And, as the same,

$$S_i < S'_i = \frac{a_{1i}}{r_i} + ... + \frac{a_{ii} r_i}{r_i} + ... + \frac{a_{ni}}{r_i} < 1$$

at the end, we have,

$$\|\hat{\bar{r}} A \hat{\bar{r}}^{-1}\| = \sup_j \{S'_j\} < 1$$

Given that all the industries directly use the product i, when increasing the size of the unit i a decrease in all S_j, $j \neq I$, is caused.

If $a_{ij} = 0$, it means that the industry j does not directly use the product i but it is indirectly used, as the system is indecomposable. Let´s consider, then, the complete circuit:

$$i = i\,(1) \rightarrow i\,(2) \rightarrow \ldots \rightarrow i\,(t) \rightarrow i\,(t{+}1) \rightarrow \ldots \rightarrow i\,(k) = i$$

in which $i\,(t{+}1)$ is the first vertex for which $S_{i\,(t+1)} = 1$.

Now, let´s take $r_{i(t)}$, such that $S_{i(t)} < r_{i(t)} < 1$ and that $\bar{r}_{i(t)}$ is the vector whose components, except the one from order $i\,(t)$, are equal to $r_{i(t)}$. The industry $i\,(t{+}1)$ directly uses the product $i\,(t)$, so that the addition of the column elements $i\,(t{+}1)$ of

$$A' = \hat{\bar{r}}_{i(t)} \cdot A \cdot \hat{\bar{r}}_{i(t)}^{-1}$$

will be

$$S'_{i(t+1)} = a_{1,i(t+1)} + \ldots + a_{i(t),i(t+1)}\, r_{i(t)} + \ldots + a_{n,i(t+1)} < S_{i(t+1)} = 1$$

On the other side, given the election of $r_{i\,(t)}$, we also have $S'_k < 1$, $k \leq i\,(t)$. We can restart the process from $i\,(t{+}1)$, obtaining real numbers $r_{i(t)}, r_{i(t+1)}, \ldots, r_{i(t+p)}$.

Being r the vector of R^n whose components , except the one from order $i\,(t)$, are equal to $r_{i(t)}$, those of order $i\,(t{+}1)$ will be equal to $r_{i(t+1)}\ldots$

Through elemental calculations we have that

$$\hat{r} = \hat{\bar{r}}_{i(t)} \cdot \hat{\bar{r}}_{i(t+1)} \cdot \ldots \cdot \hat{\bar{r}}_{i(t+p)}$$

and $A' = \hat{r}\, A\, \hat{r}^{-1}$ is such that

$$\|A'\| = \sup_j \{S'_j\} < 1$$

So that $I - A'$ is inversible, being its inverse positive, from where we deduce that the eigevalues of A' (and A) are strictly less than 1.

On the other side,

$$(I - A')^{-1} = \left(I - \hat{r}\, A\, \hat{r}^{-1}\right)^{-1} = \left[\hat{r}\,(I - A)\,\hat{r}^{-1}\right]^{-1} = \hat{r}\,(I - A)^{-1}\;\hat{r}^{-1}$$

For simplicity, let´s denote A instead of A'

$$(I - A)^{-1} = (\alpha_{ij}) \quad \text{y} \quad r_j = \sum_i \alpha_{ij}$$

We have, then,

$$\left\|(I - A)^{-1}\right\| = \sup_j \{r_j\}$$

If we consider the equations system $r\ (I - A) = \overline{1}$, or, what it is equivalent, $r = \overline{1}\,(I - A)^{-1}$, its solution is no other than $r = (r_1,..., r_j,..., r_n)$, where $r_j = \sum_i \alpha_{ij}$ and, as $(I - A)^{-1} = I + A + \ldots + A^n + \ldots$ the different r_j are strictly greater than 1.
From $r\,(I - A) = \overline{1}$, we have that $\forall j,\ 1 \le j \le n$

$$-\,a_{1j}\, r_1 - a_{2j}\, r_2 - \ldots + (1 - a_{jj})\, r_j - \ldots - a_{nj}\, r_n = 1$$

from where we obtain

$$1 = \frac{1}{r_j} + a_{1j}\frac{r_1}{r_j} + \ldots + a_{jj}\frac{r_j}{r_j} + \ldots + a_{nj}\frac{r_n}{r_j} \qquad (6)$$

Saying

$$A(1) = \hat{r}\, A\, \hat{r}^{-1} \quad \text{y} \quad S_j(1) = \sum_i a_{ij}(1)$$

we have that $\forall j,\ 1 \le j \le n$

$$1 = \frac{1}{r_j} + S_j \ (1)$$

If all r_j are equal, that is to say, if the sum of the column elements of $(I - A)^{-1}$ is equal to r, we will have that the sum of the columns of A is equal to $\bar{a}$, and that $1 = \frac{1}{r} + \bar{a}$, from where $r = \frac{1}{1-\bar{a}}$, being, then, $\bar{a}$ the maximum eigenvalue of A.

From the main approach of the Leontief system, $Q - A\, Q = \beta$, and being $I - A$ inversible, $Q = (I - A)^{-1}\, \beta$:

$$Q_i = \alpha_{i1}\, \beta_1 + ... + \alpha_{ij}\, \beta_j + ... + \alpha_{in}\, \beta_n$$

If we increase the industry´s surplus in one unit j, it must happen an increase $\Delta\, Q_i$ in the production of the industry i such that

$$Q_i + \Delta^j\, Q_i = \alpha_{i1}\, \beta_1 + ... + \alpha_{ij}\, (\beta_{j+1}) + ... + \alpha_{in}\, \beta_n$$

so that $\Delta^j\, Q_i = \alpha_{ij}$ represents the direct and indirect increase in the production of i following an increase of an unit of j, and being the system indecomposable, $\Delta^j\, Q_i = \alpha_{ij} \neq 0$.

The sums

$$r_j = \sum_i \alpha_{ij} = \sum_i \Delta^j\, Q_i$$

represent heterogeneous quantities of the goods that the system must produce when β_j increases in one unit. There it is the importance of considering the size of this unit relative to the size of the other units. So that a small r_i means that the unit is small relative to the different sizes.

As in the columns of $(I-A)^{-1}$ there are heterogeneous quantities of the goods directly and indirectly used by each industry, we suggest the possibility of using the sums

$$r_j = \sum_i \alpha_{ij}$$

as the elements of the original system transformed into another, structurally equivalent, one.

Let´s denote $\widetilde{A} = (\alpha_{ij}) = (I-A)^{-1}$, and suppose it exists at least one j such that

$$\sum_i \alpha_{ij} < \left\| \widetilde{A} \right\|$$

From $I - \hat{r}\, A\, \hat{r}^{-1} = \hat{r}\,(I-A)\,\hat{r}^{-1}$ we also have

$$\left(I - \hat{r}\, A\, \hat{r}^{-1}\right)^{-1} = \hat{r}\,(I-A)^{-1}\,\hat{r}^{-1} = \hat{r}\,\widetilde{A}\,\hat{r}^{-1}$$

Given that by construction

$$r_j = \sum_i \alpha_{ij}$$

the following matrix

$$\left(\beta_{ij}\right) = (I-A)^{-1}\,\hat{r}^{-1}$$

is such that $\forall j,\ 1 \le j \le n$

$$\sum_i \beta_{ij} = 1$$

And the sum of each column j of

$$\hat{r}\left(\beta_{ij}\right) = \hat{r}\,\widetilde{A}\,\hat{r}^{-1}$$

is

$$\inf_i \{r_i\}\left(\sum_i \beta_{ij}\right) < \sum_i r_i\,\beta_{ij} < \sup_i \{r_i\}\left(\sum_i \beta_{ij}\right) = \sup_i \{r_i\} = \left\|\widetilde{A}\right\|$$

from where

$$\inf_i \{r_i\} < \left\|\hat{r}\,\widetilde{A}\,\hat{r}^{-1}\right\| < \sup_i \{r_i\} < \left\|\widetilde{A}\right\|$$

being the rule of the transformed matrix strictly minor than the rule of the original one. Let´s say, then, that

$$\begin{aligned}
&\widetilde{A}\,(1) = \hat{r}\,\widetilde{A}\,\hat{r}^{-1}\\
&\hat{r}\,(1) = \overline{1}\,\widetilde{A}\,(1)\\
&\underline{\alpha}\,(1) = \inf_i \{r_i\,(1)\}\\
&\overline{\alpha}\,(1) = \sup_i \{r_i\,(1)\} = \left\|\widetilde{A}_1\right\|
\end{aligned}$$

Through an identical reasoning, we have

$$\underline{\alpha}\,(1) < \left\|\hat{r}\,(1)\;\;\widetilde{A}\,(1)\;\;\hat{r}\,(1)^{-1}\right\| < \overline{\alpha}\,(1)$$

Given that $\hat{r}\,(1) = \overline{1}\,\hat{r}\,\widetilde{A}\,\hat{r}^{-1}$, we have that $\forall j,\;\; 1 \le j \le n$

$$r_j\,(1) = \frac{1}{r_j}\left(\alpha_{1j}\,r_1 + \ldots + \alpha_{nj}\,r_n\right) > \inf_i \{r_i\}\left(\frac{\alpha_{1j} + \ldots + \alpha_{nj}}{r_j}\right) = \inf_i \{r_i\}$$

so that,

$$\underline{\alpha}\,(1) > \underline{\alpha}\,(0) = \inf_i \{r_i\} \quad \text{and} \quad \overline{\alpha}\,(1) < \overline{\alpha}\,(0) = \left\|\widetilde{A}\right\|$$

Repeating the reasoning we obtain a matrix series

$$\widetilde{A}\,(0) = \widetilde{A}\;\;,\;\;\widetilde{A}\,(1)\;\;,\ldots,\;\;\widetilde{A}\,(n)\;\;,\ldots$$

and two series of real numbers: $(\underline{\alpha}(n))$ strictly increasing and upper-bounded and thus convergent: $\underline{\alpha} = \lim \underline{\alpha}(n)$; and $(\overline{\alpha}(n))$ strictly decreasing and lower-bounded, and thus also convergent: $\overline{\alpha} = \lim \overline{\alpha}(n)$.

Now we will demonstrate that being $\widetilde{A}$ indecomposable, necessarily $\overline{\alpha} = \underline{\alpha}$. By construction, the different $\widetilde{A}(n)$ are indecomposable, and the change of $\widetilde{A}(n)$ into $\widetilde{A}(n+1) = \hat{r}(n)\ \widetilde{A}(n)\ \hat{r}(n)^{-1}$ is done by jointly readjusting the size of the different units, as each product directly and indirectly goes into the production of the other products.

We will demonstrate that necessarily

$$\underline{\alpha} = \lim \underline{\alpha}(n) = \overline{\alpha} = \lim \overline{\alpha}(n)$$

Firstly, let´s observe that with a given vector $r \in \mathrm{R}^n$ of strictly positive components,

$$\|\hat{r}\| = \sup_i \{r_i\}$$
$$\|\hat{r}^{-1}\| = \sup_i \left\{\frac{1}{r_i}\right\} = \frac{1}{\inf_i \{r_i\}}$$

from where

$$\|\hat{r}^{-1}\| = \|\hat{r}\|^{-1} = \frac{1}{\|\hat{r}\|} \quad \text{if and only if} \quad \frac{1}{\inf_i \{r_i\}} = \frac{1}{\sup_i \{r_i\}}$$

That is to say, only if all the r_i are equal.

Then,

$$\overline{\alpha} = \lim\ \overline{\alpha}(n+1) = \lim \left\|\widetilde{A}(n+1)\right\| = \lim\ \left\|\hat{r}(n)\cdot \widetilde{A}(n)\cdot \hat{r}(n)^{-1}\right\| \leq$$
$$\leq \lim \left[\|\hat{r}(n)\|\cdot \left\|\widetilde{A}(n)\right\| . \left\|\hat{r}(n)^{-1}\right\|\right]$$

because M_n (R) is a Banach algebra, and the rule of a matrix product is minor or equal to the rules´product, and given that,

$$\lim \left\| \widetilde{A}\,(n) \right\| = \lim \overline{\alpha}\,(n) = \overline{\alpha}$$

we must have

$$\lim \left\| \hat{r}\,(n)^{-1} \right\| = \lim \left\| \hat{r}\,(n) \right\|^{-1}$$

That is to say,

$$\frac{1}{\underline{\alpha}} = \ (\overline{\alpha})^{-1}$$

or, because it is the same, $\underline{\alpha} = \overline{\alpha}$ and then,

$$\widetilde{A}' = \lim\ \widetilde{A}\,(n) = (\alpha'_{ij})$$

is such that $\forall j,\ 1 \leq j \leq n$

$$\sum_i \alpha'_{ij} = \overline{\alpha}$$

We have then, $\overline{\alpha}$ maximum eigenvalue of

$$\widetilde{A}' = \lim\ \widetilde{A}\,(n)$$

and given that, by construction, setting

$$A\,(n+1) = \hat{r}\,(n)\ \ A\,(n)\ \ \hat{r}\,(n)^{-1}$$

we also have

$$\left(I - A(n)\right)^{-1} = \widetilde{A}(n)$$

Inverting both sides, it results

$$A(n) = I - \widetilde{A}(n)^{-1}$$

so that

$$A' = \lim A(n) = \lim \left(I - \widetilde{A}(n)^{-1}\right) = I - \widetilde{A}'^{-1}$$

and the maximum eigenvalue $\overline{a}$ of A' is

$$\overline{a} = 1 - \frac{1}{\overline{\alpha}}$$

Furthermore, and also by construction, matrices $\widetilde{A}' = \left(\alpha'_{ij}\right)$ and $A' = (a_{ij})$ are such that $\forall j,\ 1 \leq j \leq n$

$$\sum_i \alpha'_{ij} = \overline{\alpha} \quad \text{y} \quad \sum_i a_{ij} = \overline{a}$$

That´s because they admit as positive eigenvector on the left $\overline{1} = (1,...,1)$ or any positive multiple of it.

The economic interpretation of the procedure is easier to see now. Each column j of $\widetilde{A}'$ represents heterogeneous quantities of the system´s goods needed to produce an unit of j as final good, and given that all the columns sum $\overline{\alpha}$, in order to produce an additional quantity of *any* good, we need the same numerical quantity of the system´s heterogeneous goods, so that an unit of good i must be equivalent to another of good j, i.e. all the units can be expressed in terms of one of them –or a multiple or part of it– chosen as numeraire.

With the target of illustrating the consecutive approximation procedure, let´s take again the simple example of Sraffa relative to the wheat and iron industry, in which the wheat unit is the quarter and the iron unit is the ton.

The technological matrix was

$$A = \begin{pmatrix} 0{,}486956 & 6 \\ 0{,}020869 & 0{,}4 \end{pmatrix}$$

For which $S_1(0) \approx 0{,}507825$ and $S_2(0) = 6{,}4$, as the quarter is an excessively small-sized unit in comparison with the ton (or the ton excessively larger than the quarter).

Leontief´s inverse:

$$\widetilde{A}(0) = (I - A)^{-1} \cong \begin{pmatrix} 3{,}285654 & 32{,}856548 \\ 0{,}11428 & 2{,}809475 \end{pmatrix}$$

from where $r_1(0) \cong 3{,}4$ and $r_2(0) \cong 35{,}666023$.

If we say now

$$\widetilde{A}(1) = \hat{r}(0)\ (I - A)^{-1}\ \hat{r}(0)^{-1}$$

we obtain

$$\widetilde{A}(1) \cong \begin{pmatrix} 3{,}285654 & 3{,}1321788 \\ 1{,}198794 & 2{,}809475 \end{pmatrix}$$

from where $r_1(1) = 4{,}484448$ and $r_2(1) = 5{,}941653$ and

$$\widetilde{A}(2) = \hat{r}(1)\ \widetilde{A}(1)\ \hat{r}(1)^{-1} \cong \begin{pmatrix} 3{,}285654 & 2{,}364003 \\ 1{,}588323 & 2{,}809475 \end{pmatrix}$$

where $r_1(2) = 4{,}873977$ and $r_2(2) = 5{,}173478$.

In the following stage we would obtain r_1 (3) = 4,971577 and r_2 (3) = 5,036622. We can see that the series $\underline{\alpha}(n) = r_1(n)$ and $\overline{\alpha}(n) = r_2(n)$ tend to $\underline{\alpha} = \overline{\alpha}$ = 5, and that the maximum eigenvalue of matrix A' converted into A would be $\overline{a} = 1 - \frac{1}{5} = 0{,}8$, as we expected.

We can also observe that in this case of two unique industries

$$\frac{r_2(0) \cdot r_2(1) \cdot r_2(2) \cdot r_2(3)}{r_1(0) \cdot r_1(1) \cdot r_1(2) \cdot r_1(3)} \cong \frac{5521{,}84}{369{,}45} = 14{,}94 \cong 15$$

That was the exchange relation in the original system:

1 iron ton = 15 wheat quarters

HOMOGENEIZATION OF THE UNITS

We have demonstrated that if the economic system has any kind of surplus and it is irreductible, the máximum eigenvalue of A, $\overline{a}$, is strictly minor than 1 and the system can be transformed into another equivalent where its technological matrix $A' = (a'_{ij})$ is such that $\forall j,\ 1 \le j \le n$

$$\sum_i a'_{ij} = \overline{a}$$

Being $\overline{1} = (1,\ldots,1)$ eigenvector on the left of A'

$S \in \mathrm{R}^n,\ S > 0\ \ (\forall i,\ S_i > 0)$, and denoting $\frac{1}{S}$ vector $\left(\frac{1}{S_i}\right)_i$

matrix $A(1) = \hat{S}\, A'\, \hat{S}^{-1}$ has the same eigenvalues as A' and, as $A' = \hat{S}^{-1} A(1)\, \hat{S}$ and $\overline{1}\, \hat{S}^{-1} = \frac{1}{S}$, of $\overline{1}\, A' = \overline{a}\, \overline{1}$ we deduce that $\left(\overline{1}\, \hat{S}^{-1}\right) \cdot A(1) = \overline{a} \cdot \overline{1}\, S^{-1}$, i.e. $\frac{1}{S} A(1) = \overline{a}\, \frac{1}{S}$ and $\frac{1}{S}$ is an eigenvector on the left of A (1).

$A(1)$ ($S \in \mathrm{R}^n$, $S > 0$) are a family of matrices of $M_n(\mathrm{R})$ representative of the Leontief system set. In particular, given the original matrix A, it exists

$$r = \frac{1}{S} \in \mathrm{R}^n, \quad r > 0$$

such that $r\,A = \bar{a}\,r$, being r the eigenvector on the left of A.

Let´s take again system (4) and modify the size of the units in equal terms as r_i components of vector r.

From $r\,A = \bar{a}\,r$ we have that $\forall j,\ 1 \le j \le n$

$$r_1\,a_{ij} + \ldots + r_i\,a_{ij} + \ldots + r_j\,a_{jj} + \ldots + r_n\,a_{nj} = \bar{a}\,r_j \qquad (7)$$

so that

$$\frac{r_1}{r_j}a_{ij} + \ldots + \frac{r_i}{r_j}a_{ij} + \ldots + \frac{r_j}{r_j}a_{jj} + \ldots + \frac{r_n}{r_j}a_{nj} = \bar{a}$$

And setting $A' = (a'_{ij})$ with

$$a'_{ij} = \frac{r_i}{r_j}a_{ij}$$

$$A' = \hat{r}\,A\,\hat{r}^{-1}$$

and $\forall j,\ 1 \le j \le n$

$$\sum_i a'_{ij} = \bar{a}$$

DEFINITION. Given an indecomposable Leontief system, we will call homogenized system the transformed equivalent system whose technological matrix A is such than $\forall j,$

$$\sum_i a_{ij} = \bar{a}$$

and we will call A *the homogenized matrix*.

Among the infinite technological matrices $A\,(S)$ representative of the same system, it will be useful to take as representative the homogenized one, in particular when referring to the economic interpretation of the main matrix A and its Leontief inverse.
System (7) is equivalent to $r\,(\bar{a}\,I - A) = 0$ (an homogeneous system of n equations and n unknowns with range n–1 (as $\bar{a}$ is the simple root of the characteristical equation). For its resolution we can take one of the original units (or part of them) as numeraire, for example, $r_n = 1$.
We will transform the system such that all the units will be expressed in terms of $r_n = 1$, so they will behave as single product units inside the system. Then, the heterogeneous quantities that appear in each column j of A behave as if they were homogeneous.

THE LEONTIEF INVERSE

Being A homogenized, we have that $\forall j,$

$$\sum_i a_{ij} = \bar{a} = \|A\|$$

Let´s set, then, $(\gamma_{ij}) = A^2$, and we have

$$\gamma_{ij} = \sum_k a_{ik}\, a_{kj}$$

$$\sum_i \gamma_{ij} = \sum_i \sum_k a_{ik}\, a_{kj} = \sum_k a_{kj} \left(\sum_k a_{ik} \right) = \sum_k a_{kj} \cdot \bar{a} = \bar{a}^2$$

That is to say, $\|A^2\| = \|A\|^2$, from where we deduce that $\forall n \quad \|A^n\| = \|A\|^n$ and the sum of the columns of A^n are all equal, and equal to $\bar{a}^n$.
Then, as

$$(I - A)^{-1} = I + A + ... + A^n + ...$$

The sum of any column elements of $(I-A)^{-1}$ will be

$$1+\overline{a}+\ldots+\overline{a}^{n}+\ldots=\frac{1}{1-\overline{a}}$$

We have previously seen that each column j of $(I-A)^{-1}$ represents heterogeneous quantities of goods needed for the production of an additional unit of j and, as all column sums are the same in the homogenized system, this means that we need the same quantities of goods to produce an additional unit of any of them.

THE STRUCTURAL RATIO

If the system has previously been homogenized, matrix $A = (a_{ij})$ is such that $\forall j,\ 1 \le j \le n$

$$\sum_i a_{ij} = \overline{a}$$

or even

$$\sum_i q_{ij} = \overline{a}\, Q_j$$

Adding both sides over j,

$$\sum_{i,j} q_{ij} = \overline{a}\left(\sum_j Q_j\right)$$

or, if its preferred,

$$\frac{\sum_{i,j} q_{ij}}{\sum_j Q_j} = \overline{a} \qquad (8)$$

That is to say, the ratio between the total quantity of goods used in production and the total quantity of goods produced is equal to the maximum eigenvalue $\overline{a}$.

From (8), and taking into account that

$$\sum_{j} Q_j = \sum_{i,j} q_{ij} + \sum_{j} \beta_j$$

we also deduce

$$\frac{\sum_{j} \beta_j}{\sum_{\mathrm{j}} Q_j} = 1 - \overline{a} \qquad (8')$$

$$\frac{\sum_{j} \beta_j}{\sum_{i,j} q_{ij}} = \frac{1}{\overline{a}} - 1 \qquad (8'')$$

We obtain, as function of $\overline{a}$, ratios among surplus produced, and the total production of the system. We will say that (8) is the *system´s structural ratio.*

MATHEMATICAL APPENDIX

We will denote as *Mn* (R) the vectorial space of the square matrix of order n with real coefficients. In *Mn* (R) we also have another internal composition rule, the square matrix product: if $A = (a_{ij}) \in Mn$ (R) and $B = (b_{ij}) \in Mn$ (R), $A.B = (\gamma_{ij})$, with $(\gamma_{ij}) = \sum_{k} a_{ik}\, b_{kj}$. The product is associative, it is distributive over matrix addition, and it admits as neutral element the identity matrix *I.*

THEOREM 1. In *Mn* (R) if $A = (a_{ij})$, setting $\|A\| = \sup_{j} \sum_{i} |a_{ij}|$, $\|\cdot\|$ is a rule in *Mn* (R), and thus *Mn* (R) is a Banach algebra.

<u>Proof:</u>

a) That $\|A\| = \sup_{j} \sum_{i} |a_{ij}|$ is a rule in *Mn* (R) is obvious.

b) Mn (R) is complete. So, being $(A^t)_t = \left(a_{ij}^t\right)_t$ a Cauchy series in Mn (R) ,

$$\forall \varepsilon > 0, \exists\, t_0 \in \mathrm{N} \,/\; p \geq t_0,\;\; q \geq t_0 \Rightarrow \left\| A^p - A^q \right\| < \varepsilon$$

but

$$\left\| A^p - A^q \right\| < \varepsilon \Leftrightarrow \sup_j \sum_{i=1}^{n} \left| a_{ij}^p - a_{ij}^q \right| < \varepsilon$$

And in particular, being i and j fixed, we have $\left| a_{ij}^p - a_{ij}^q \right| < \varepsilon$, so that the series of real numbers $\left(a_{ij}^t\right)_t$ is Cauchy in R and converge to a real number that we will denote as a_{ij}.

Setting then, $A = (a_{ij})$, it is enough to demonstrate that $(A^t)_t$ converges to A in Mn (R).

Be $\varepsilon \in$ R, $\varepsilon>0$, as $\left(a_{ij}^t\right)_t$ converges in R to a_{ij},

$$\exists\, t\,(i, j) \in \mathrm{N} \,/\, t \geq t\,(i, j) \;\Rightarrow \left| a_{ij}^t - a_{ij} \right| < \frac{\varepsilon}{n}$$

And if we take $t_0 = \sup \left\{ t\,(i,j),\;\; 1 \leq i \leq n,\;\; 1 \leq j \leq n \right\}$, then

$$\forall t \geq t_0 \text{ y } \forall j,\; 1 \leq j \leq n,\; \sum_{i=1}^{n} \left| a_{ij}^t - a_{ij} \right| < n \cdot \frac{\varepsilon}{n} = \varepsilon$$

From where it results $\sup_j \left(\sum_{i=1}^{n} \left| a_{ij}^t - a_{ij} \right| \right) < \varepsilon$ or even $\left\| A^t - A \right\| < \varepsilon$, so that $(A^t)_t$ converges to A. Mn (R) is a Banach space.

c) $\forall\, A, B \in Mn$ (R), $\left\| A \cdot B \right\| \leq \left\| A \right\| \cdot \left\| B \right\|$ and $\left\| I \right\| = 1$, if $A = (a_{ij})$ and $B = (b_{ij})$, setting $A.B = (\gamma_{ij})$, with $(\gamma_{ij}) = \sum_k a_{ik}\, b_{kj}$ we have:

$$\left\| A \cdot B \right\| = \left\| (\gamma_{ij}) \right\| = \sup_j \left(\sum_i \left| \gamma_{ij} \right| \right) = \sup_j \left(\sum_i \left| a_{ik}\, b_{kj} \right| \right) \leq \sup_j \sum_i \sum_k \left| a_{ik}\, b_{kj} \right| \leq$$

$$\leq \sup_j \left[\sum_i \left| b_{ij} \right| \left(\sum_k \left| a_{ki} \right| \right) \right] \leq \left\| A \right\| .\; \sup_j \sum_i \left| b_{ij} \right| = \left\| A \cdot B \right\|$$

The chosen rule is “compatible” with the matrix product. So it is obvious that $\left\| I \right\| = 1$.

THEOREM 2. For $A \in Mn$ (R) to be invertible in Mn (R), it is necessary and sufficient that it exists $D \in Mn$ (R), D inversible and such that $\left\| I - A\, D^{-1} \right\| = K < 1$.

Proof: If A is inversible, we take $D = A$, we have $\left\| I - A \cdot A^{-1} \right\| = 0 < 1$. In both sides, we suppose that it exists D inversible, such that $\left\| I - A D^{-1} \right\| = K < 1$, the application

$$\Phi : Mn\ (\mathrm{R}) \to Mn\ (\mathrm{R}) :$$
$$\phi\ (x) = D^{-1} + x\ \left(I - A D^{-1}\right)$$

is such that

$$\left\| \phi (x) - \phi (x') \right\| = \left\| (x - x') \left(I - A D^{-1}\right) \right\| \le \left\| x - x' \right\| \cdot \left\| I - A D^{-1} \right\|$$

(because Mn (R) is a Banach algebra), and as $\left\| I - A D^{-1} \right\| = K < 1$, Φ is a complete contraction in Mn (R), and admits an unique fixed point $\bar{x}$.

We have $\phi\left(\bar{x}\right) = \bar{x}$ or

$$D^{-1} + \bar{x}\left(I - A D^{-1}\right) = \bar{x}\ ,\ \ \bar{x} A\, D^{-1} = D^{-1},$$

and multiplying by D on the right, $\bar{x} A = I$ and $\bar{x}$ is the inverse of A.

Moreover, $\forall\ x_0 \in Mn$ (R), the series $x_0,\ x_1 = \phi\ (x_0), \dots,\ x_n = \phi\ (x_{n-1}), \dots$ converges at $\bar{x} = A^{-1}$ (fixed point theorem).

Definition. Being $A = (a_{ij}) \in Mn$ (R), we will say that A is column dominant diagonal if $\forall\ j,\ \ 1 \le j \le n,\ \ \left|a_{jj}\right| > \sum_{i \ne j} \left|a_{ij}\right|$; we will also say that A is *Leontief* if

$$\begin{cases} \forall i, \quad a_{ii} > 0 \\ a_{ij} \le 0 \ \text{ si } \ i \ne j \end{cases}$$

Obviously, if A is the technological matrix in a Leontief system, $I - A$ is Leontief.

THEOREM 3. If $A = (a_{ij}) \in Mn$ (R) if Leontief´s column dominant diagonal, A is inversible in Mn (R) and also, $A^{-1} \ge 0$.

Proof. Being A dominant diagonal, $\forall j$, $1 \le j \le n$,

$$a_{jj} > \sum_{i \ne j} \left|a_{ij}\right| \ \ \text{y} \ \ \frac{1}{a_{jj}} \sum_{i \ne j} \left|a_{ij}\right| = k < 1$$

Let´s consider the matrix D defined by $d_{jj} = a_{jj}$, $d_{ij} = 0$ if $i \ne j$. As $d_{jj} = a_{jj} > 0$, D is inversible in Mn (R) and also,

$$\left\| I - A D^{-1} \right\| = \sup_j \left[\frac{1}{a_{jj}} \left(\sum_{i \ne j} \left|a_{ij}\right| \right) \right] \le k < 1$$

so that the application

$$\Phi : Mn\,(\mathrm{R}) \to Mn\,(\mathrm{R}) :$$
$$\phi\,(x) = D^{-1} + x\,\left(I - A\,D^{-1}\right)$$

is a contraction in Mn (R) and it admits an unique fixed point $\bar{x} = A^{-1}$. Moreover, given that $D^{-1} \geq 0$ and $\left(I - A\,D^{-1}\right) \geq 0$, starting from $x_0 \geq 0$, the row x_0, $x_1=\phi\,(x_0)$,..., $x_n=\phi\,(x_{n-1})$,... is such that $\forall n$, $x_n \geq 0$ and, consequently, $\lim\limits_{n\to\infty} x_n = \bar{x} = A^{-1} \geq 0$.

THEOREM 4. If $A \in Mn$ (R) and λ is eigenvalue of A, we have $|\lambda| \leq \|A\|$.

Proof. If λ is eigenvalue of A, $\lambda I - A$ is not inversible and then, (theorem 2), for the whole inversible matrix D we have $\left\|I - (\lambda\,I - A)D^{-1}\right\| \geq 1$. If $\lambda = 0$, this means that A is not inversible and, obviously, $|\lambda| = 0 \leq \|A\|$. If $\lambda \neq 0$, and considering the diagonal matrix $D = (d_{ij})$ defined by $d_{ii}=\lambda$, $d_{ij} = 0$ if $i\neq j$, we have:

$$\left\|I - (\lambda\,I - A)D^{-1}\right\| = \sup_j \left[\frac{1}{\lambda}\left(\sum_i^n |a_{ij}|\right)\right] = \frac{1}{\lambda}\|A\|$$

And as it must be $\frac{1}{|\lambda|}\|A\| \geq 1$, it results $|\lambda| \leq \|A\|$.

THEOREM 5. Being $A \in Mn$ (R), and $A \geq 0$

a) If $\forall\, j$, $1 \leq j \leq n$, $\sum\limits_{i=1}^{n} a_{ij} = \bar{a} = \|A\|$, then $\bar{a}$ is maximum real eigenvalue of A that admits $\bar{1} = (1,...,1)$ as associated eigenvector on the left.

b) If $\lambda \in$ R is such that $\lambda > \|A\|$, then $\lambda\,I - A$ is inversible and $(\lambda\,I - A)^{-1} \geq 0$.

Proof.

a) It results that $\bar{1}\,A = \bar{a}\,\bar{1}$, and $\bar{a}$ is a real eigenvalue. On the other side, as for every eigenvalue λ, $|\lambda| \leq \|A\|$ (theorem 4), it results that $\bar{a}$ is the maximum eigenvalue.

b) Let´s consider matrix $D = (d_{ij})$ defined by $d_{ii} = \lambda$ and $d_{ij} = 0$, if $i \neq j$. We have that $\left\| I - (\lambda I - A) D^{-1} \right\| = \frac{\|A\|}{\lambda} < 1$ and the application

$$\Phi : Mn\,(\mathrm{R}) \rightarrow Mn\,(\mathrm{R}) :$$
$$\phi(x) = D^{-1} + x \left(I - (\lambda I - A) D^{-1} \right)$$

is a contraction in Mn (R) that admits an unique fixed point $\bar{x} = (\lambda I - A) D^{-1}$ (theorem 2) and, given that $D^{-1} \geq 0$ and $I - (\lambda I - A) D^{-1} \geq 0$, starting from $x_0 \geq 0$, the row x_0, $x_1 = \phi(x_0)$,..., $x_n = \phi(x_{n-1})$,... is such that $\forall n \quad x_n \geq 0$ and $\bar{x} = (\lambda I - A)^{-1} = \lim \; x_n \geq 0$.

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